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ON THE THEORY OF CURVED SURFACES, AND CANONICAL SYSTEMS IN PROJECTIVE DIFFERENTIAL GEOMETRY*

BY

G. M. GREEN

In a series of five memoirs,† Wilczynski has developed a projective theory of curved surfaces, employing in his study a method which is peculiarly his own. This consists in the consideration of a certain system of partial differential equations and the theory of its invariants and covariants. The analysis, however, is often extremely complicated, so that any simplification of the problem is welcome. In the following pages will be indicated a procedure which, though carried out in detail for the particular problem of the theory of surfaces only, seems to be of very general applicability.

In the theory of curved surfaces, a very natural plan presents itself, viz., to assume that the surface is referred to its asymptotic curves. This is done by Wilczynski in his first three memoirs, and on this assumption he calculates the complete system of invariants and covariants. But the determination of the asymptotic lines on a surface requires the integration of a quadratic partial differential equation of the first order, and it is in general impossible to carry out this integration explicitly. There is, then, an apparent restriction in generality in supposing known the asymptotic lines. To remove this objection, Wilczynski, in his fourth memoir, started to set up a system of invariants for the most general differential equations of the unrestricted problem. Notwithstanding the complications involved in the calculations, the method and the results are very elegant. The question of the completeness of the system of invariants is, however, left open.

The invariants of the first three memoirs are functions of the coefficients of a canonical system, i. e., of a system of differential equations whose independent variables are parameters of the asymptotic lines of the surface. Their invariantive property holds with respect to all those transformations of the independent variables which preserve the asymptotic lines as para-

* Presented to the Society under a different title, December 31, 1913.

† *Projective differential geometry of curved surfaces.* These Transactions, vols. 8–10 (1907–1909). We shall have occasion to refer to the *First Memoir*, vol. 8 (1907), pp. 233–260, and the *Fourth Memoir*, vol. 10 (1909), pp. 176–200.

meter curves. The invariants of the fourth memoir, on the other hand, are functions of the coefficients of an arbitrary system, and their invariantive property holds with reference to arbitrary transformations of the independent variables. It is clear, however, from the geometric significance of the invariants, as well as on general invariant- and group-theoretic grounds, that the two complete systems of invariants must be equivalent. In fact, the invariants of the first three memoirs are merely the canonical forms which those of the fourth memoir assume, when the general system of partial differential equations there considered is supposed to be reduced to its canonical form.

In the present paper, we propose to show how a complete system of concomitants for the general theory of surfaces may be constructed. The method employed enables us to avail ourselves of all the simplifications which would be possible if the asymptotic curves were known; in fact, we shall show that there is no loss of generality whatever in supposing, as Wilczynski does in his first three memoirs, that the surface is referred to its asymptotic curves.

The application of the principles developed in this paper to any problem in projective differential geometry will be best understood after their application to the theory of curved surfaces has been developed in detail. We therefore proceed at once to the consideration of the differential equations for this particular problem.

The most general system of two linear homogeneous partial differential equations of the second order in one dependent and two independent variables is of the form

$$\begin{aligned} &Ay_{uu} + By_{uv} + Cy_{vv} + Dy_u + Ey_v + Fy = 0, \\ &A'y_{uu} + B'y_{uv} + C'y_{vv} + D'y_u + E'y_v + F'y = 0. \end{aligned} \tag{1}$$

In all cases which are of interest in the theory of curved surfaces, system (1) may be reduced to one of the form

$$\begin{aligned} y_{uu} &= a\,y_{uv} + b\,y_u + c\,y_v + d\,y, \\ y_{vv} &= a'y_{uv} + b'y_u + c'y_v + d'y, \end{aligned} \tag{2}$$

where $aa' - 1$ is not identically equal to zero, the reduction requiring only operations which are always practically possible.* Our object is to set up a complete system of invariants and covariants for system (2), in the sense in which these terms are used by Wilczynski.

Let us, with Wilczynski, transform the independent variables in (2) by the transformation

$$\bar{u} = \phi(u, v), \qquad \bar{v} = \psi(u, v). \tag{3}$$

* E. J. Wilczynski, *First Memoir*, p. 234.

We find

$$y_u = \phi_u \bar{y}_u + \psi_u \bar{y}_v, \qquad y_v = \phi_v \bar{y}_u + \psi_v \bar{y}_v, \tag{4}$$

$$\begin{aligned} y_{uu} &= \phi_u^2 \bar{y}_{uu} + 2\phi_u \psi_u \bar{y}_{uv} + \psi_u^2 \bar{y}_{vv} + \phi_{uu} \bar{y}_u + \psi_{uu} \bar{y}_v, \\ y_{uv} &= \phi_u \phi_v \bar{y}_{uu} + (\phi_u \psi_v + \psi_u \phi_v) \bar{y}_{uv} + \psi_u \psi_v \bar{y}_{vv} + \phi_{uv} \bar{y}_u + \psi_{uv} \bar{y}_v, \\ y_{vv} &= \phi_v^2 \bar{y}_{uu} + 2\phi_v \psi_v \bar{y}_{uv} + \psi_v^2 \bar{y}_{vv} + \phi_{vv} \bar{y}_u + \psi_{vv} \bar{y}_v, \end{aligned} \tag{5}$$

in which $\bar{y}_u$, $\bar{y}_v$, etc., denote $\partial y/\partial \bar{u}$, $\partial y/\partial \bar{v}$, etc.

Substituting in (2), we obtain the equations

$$\begin{aligned} &(\phi_u^2 - a\phi_u \phi_v) \bar{y}_{uu} + (\psi_u^2 - a\psi_u \psi_v) \bar{y}_{vv} \\ &\qquad = [a(\phi_u \psi_v + \psi_u \phi_v) - 2\phi_u \psi_u] \bar{y}_{uv} \\ &\qquad + (b\phi_u + c\phi_v - \phi_{uu} + a\phi_{uv}) \bar{y}_u \\ &\qquad + (b\psi_u + c\psi_v - \psi_{uu} + a\psi_{uv}) \bar{y}_v + d\bar{y}, \\ &(\phi_v^2 - a' \phi_u \phi_v) \bar{y}_{uu} + (\psi_v^2 - a' \psi_u \psi_v) \bar{y}_{vv} \\ &\qquad = [a'(\phi_u \psi_v + \psi_u \phi_v) - 2\phi_v \psi_v] \bar{y}_{uv} \\ &\qquad + (b' \phi_u + c' \phi_v - \phi_{vv} + a' \phi_{uv}) \bar{y}_u \\ &\qquad + (b' \psi_u + c' \psi_v - \psi_{vv} + a' \psi_{uv}) \bar{y}_v + d' \bar{y}. \end{aligned} \tag{6}$$

If these be solved for $\bar{y}_{uu}$ and $\bar{y}_{vv}$, one obtains a system of the same form as (2); denoting its coefficients by $\bar{a}$, $\bar{b}$, etc., we find that

$$\begin{aligned} \bar{a} &= \frac{a\psi_v^2 - 2\psi_u \psi_v + a' \psi_u^2}{\phi_u \psi_v + \phi_v \psi_u - a\phi_v \psi_v - a' \phi_u \psi_u}, \\ \bar{a}' &= \frac{a\phi_v^2 - 2\phi_u \phi_v + a' \phi_u^2}{\phi_u \psi_v + \phi_v \psi_u - a\phi_v \psi_v - a' \phi_u \psi_u}. \end{aligned} \tag{7}$$

We may therefore make $\bar{a}$ and $\bar{a}'$ vanish by taking for ϕ and ψ solutions of the two factors of the differential equation

$$a\chi_v^2 - 2\chi_u \chi_v + a' \chi_u^2 = 0,$$

i. e., by choosing ϕ and ψ so as to satisfy the differential equations*

$$(1 + \sqrt{1 - aa'})\phi_u - a\phi_v = 0, \qquad (1 - \sqrt{1 - aa'})\psi_u - a\psi_v = 0, \tag{8}$$

which are distinct, since $aa' - 1 \neq 0$.

Suppose this to have been done. The new system will take the form

$$\begin{aligned} \bar{y}_{uu} &= \bar{b}\ \bar{y}_u + \bar{c}\ \bar{y}_v + \bar{d}\ \bar{y}, \\ \bar{y}_{vv} &= \bar{b}'\ \bar{y}_u + \bar{c}'\ \bar{y}_v + \bar{d}'\ \bar{y}. \end{aligned} \tag{9}$$

* *First Memoir*, p. 243, where it is also proved that the denominator in equations (7) will not vanish identically.

It is for this system of differential equations that Wilczynski, in his *First Memoir*, calculates a complete system of invariants and covariants. But the reduction of system (2) to the form (9)—which is equivalent to the determination of the asymptotic curves of the surface defined by any fundamental system of solutions of (2)—requires the integration of equations (8). In general an explicit integration of (8) is of course impossible, so that apparently a theory based on system (9) is less general in form than one based on system (2). Regarded geometrically, however, the projective theory of a surface is identical with that of its asymptotic curves; for any property of the one, there must be a corresponding property of the other. It is to be expected therefore, that the invariants and covariants as calculated for system (9) must suffice to express all projective properties of the integrating surfaces of (2). The question to be answered, then, is whether the invariants and covariants of equations (9) are expressible *explicitly and in a simple manner* in terms of the coefficients of equations (2), so that the integration of equations (8) may be rendered unnecessary. We shall see that this is indeed possible, so that the identity, geometrically, of the theory of a surface and that of its asymptotic lines has its complete analytic analogue.

If we put

$$\mu = \sqrt{1 - aa'}, \qquad \mu \not\equiv 0, \tag{10}$$

equations (8) become

$$(1+\mu)\phi_u - a\phi_v = 0, \qquad (1-\mu)\psi_u - a\psi_v = 0,$$

or, multiplying the second of these by $1+\mu$ and dividing out the factor a* which appears because $1-\mu^2 = aa'$, we obtain the more symmetrical equations

$$(1+\mu)\phi_u - a\phi_v = 0, \qquad (1+\mu)\psi_v - a'\psi_u = 0.$$

By means of these, we may express ϕ_v in terms of ϕ_u, and ψ_u in terms of ψ_v:

$$\phi_v = \frac{1+\mu}{a}\phi_u, \qquad \psi_u = \frac{1+\mu}{a'}\psi_v. \tag{11}$$

By differentiation, we find also

$$\begin{aligned} \phi_{uv} &= \left(\frac{1+\mu}{a}\right)_u \phi_u + \frac{1+\mu}{a}\phi_{uu}, \qquad \psi_{uv} = \left(\frac{1+\mu}{a'}\right)_v \psi_v + \frac{1+\mu}{a'}\psi_{vv}, \\ \phi_{vv} &= \left(\frac{1+\mu}{a}\right)_v \phi_u + \frac{1+\mu}{a}\phi_{uv}, \qquad \psi_{uu} = \left(\frac{1+\mu}{a'}\right)_u \psi_v + \frac{1+\mu}{a'}\psi_{uv}, \end{aligned} \tag{12}$$

so that the second derivatives of ϕ are expressible in terms of ϕ_u and ϕ_{uu}, and those of ψ in terms of ψ_v and ψ_{vv}. Substituting in (6), we obtain the equations

* The special cases $a = 0$ or $a' = 0$ are easily discussed and may therefore be excluded from the present consideration.

$$
\begin{aligned}
&-\mu\phi_u^2\,\bar{y}_{uu}+\frac{\mu(1+\mu)^2}{a'^2}\psi_v^2\,\bar{y}_{vv}\\
&\quad=\left[\left\{b+\frac{1+\mu}{a}c+a\left(\frac{1+\mu}{a}\right)_u\right\}\phi_u+\mu\phi_{uu}\right]\bar{y}_u\\
&\qquad+\left[\left\{\frac{1+\mu}{a'}b+c-\left(\frac{1+\mu}{a'}\right)_u\right\}\psi_v-\frac{\mu(1+\mu)}{a'}\psi_{uv}\right]\bar{y}_v+d\bar{y},\\
&\frac{\mu(1+\mu)^2}{a^2}\phi_u^2\,\bar{y}_{uu}-\mu\psi_v^2\,\bar{y}_{vv}\\
&\quad=\left[\left\{b'+\frac{1+\mu}{a}c'-\left(\frac{1+\mu}{a}\right)_v\right\}\phi_u-\frac{\mu(1+\mu)}{a}\phi_{uv}\right]\bar{y}_u\\
&\qquad+\left[\left\{\frac{1+\mu}{a'}b'+c'+a'\left(\frac{1+\mu}{a'}\right)_v\right\}\psi_v+\mu\psi_{vv}\right]\bar{y}_v+d'\,\bar{y}.
\end{aligned}
\tag{13}
$$

Solving these equations for $\bar{y}_{uu}$ and $\bar{y}_{vv}$, we obtain a system of form (9), which we may write*

$$
\begin{aligned}
&\bar{y}_{uu}+2\alpha\;\bar{y}_u+2\beta\;\bar{y}_v+\gamma\;\bar{y}=0,\\
&\bar{y}_{vv}+2\alpha'\,\bar{y}_u+2\beta'\,\bar{y}_v+\gamma'\,\bar{y}=0.
\end{aligned}
\tag{14}
$$

The coefficients in these equations are of the form

$$
\begin{aligned}
&\alpha=\frac{1}{\phi_u}(\{\alpha\}+\tfrac{1}{2}\xi), \qquad \beta=\frac{\psi_v}{\phi_u^2}\{\beta\}, \qquad \gamma=\frac{1}{\phi_u^2}\{\gamma\},\\
&\alpha'=\frac{\phi_u}{\psi_v^2}\{\alpha'\}, \qquad \beta'=\frac{1}{\psi_v}(\{\beta'\}+\tfrac{1}{2}\eta), \qquad \gamma'=\frac{1}{\psi_v^2}\{\gamma'\},
\end{aligned}
\tag{15}
$$

where

$$
\xi=\frac{\phi_{uu}}{\phi_u}, \qquad \eta=\frac{\psi_{vv}}{\psi_v},
\tag{16}
$$

and $\{\alpha\}$, $\{\beta\}$, etc., are the following expressions in the coefficients of (2), entirely free from ϕ and ψ:

$$
\begin{aligned}
\{\alpha\}=&-\frac{(1-\mu)^2}{8\mu^2}\left\{b+\frac{1+\mu}{a}c+\frac{a(1-3\mu)}{(1-\mu)^2}\left(\frac{1+\mu}{a}\right)_u\right.\\
&\left.+\frac{(1+\mu)^2}{a'^2}\left[b'+\frac{1+\mu}{a}c'-\left(\frac{1+\mu}{a}\right)_v\right]\right\},\\
\{\beta\}=&-\frac{(1-\mu)^2}{8\mu^2}\left\{\frac{1+\mu}{a'}b+c-\left(\frac{1+\mu}{a'}\right)_u+\frac{1+\mu}{a'}\left(\frac{1+\mu}{a'}\right)_v\right.\\
&\left.+\frac{(1+\mu)^2}{a'^2}\left(\frac{1+\mu}{a'}b'+c'\right)\right\},
\end{aligned}
$$

* These are the same as equations (27) of Wilczynski's *First Memoir*, except that we have used Greek letters for corresponding Roman letters to avoid confusion.

$$
\begin{aligned}
\{\alpha'\} &= -\frac{(1-\mu)^2}{8\mu^2}\left\{\frac{(1+\mu)^2}{a^2}\left(b+\frac{1+\mu}{a}c\right)+b'+\frac{1+\mu}{a}c'\right. \\
&\qquad \left.+\frac{1+\mu}{a}\left(\frac{1+\mu}{a}\right)_u-\left(\frac{1+\mu}{a}\right)_v\right\}, \\
\{\beta'\} &= -\frac{(1-\mu)^2}{8\mu^2}\left\{\frac{(1+\mu)^2}{a^2}\left[\frac{1+\mu}{a'}b+c-\left(\frac{1+\mu}{a'}\right)_u\right]\right. \\
&\qquad \left.+\frac{1+\mu}{a'}b'+c'+\frac{a'(1-3\mu)}{(1-\mu)^2}\left(\frac{1+\mu}{a'}\right)_v\right\}, \\
\{\gamma\} &= -\frac{(1-\mu)^2}{4\mu^2}\left[d+\frac{(1+\mu)^2}{a'^2}d'\right], \\
\{\gamma'\} &= -\frac{(1-\mu)^2}{4\mu^2}\left[\frac{(1+\mu)^2}{a^2}d+d'\right].
\end{aligned}
\tag{17}
$$

The coefficients in equations (14) depend of course on ϕ and ψ, as shown by (15); to determine these coefficients, equations (8) must actually be integrated. But if, in the expressions for the fundamental invariants of (14)—to be written presently— we substitute for α, β, etc., their expressions as given by (15), we shall find that ϕ and ψ come out as factors of the form $\phi_u^k \psi_v^l$. For absolute invariants, $k = l = 0$. The quantities α' and β are relative invariants of system (14); the vanishing of either expresses the condition that a corresponding family of asymptotic curves of the surface consists of straight lines. The vanishing of $\{\alpha'\}$ or $\{\beta\}$, as given by (17), expresses the same condition in terms of the coefficients of system (2). The product, $\{\alpha'\}\{\beta\}$, is essentially the invariant θ of Wilczynski's *Fourth Memoir*, whose vanishing is the condition that the surface be ruled.

Besides β and α', there are two more invariants, which form with these a fundamental system. If we put

$$
f = \gamma - \alpha_{\bar{u}} - \alpha^2 - 2\beta\beta', \qquad g = \gamma' - \beta'_{\bar{v}} - \beta'^2 - 2\alpha\alpha', \tag{18}
$$

then the quantities

$$
\begin{aligned}
h &= \beta^2\left[f+\beta_{\bar{v}}-\frac{1}{4}\frac{\partial}{\partial\bar{u}}\left(\frac{\beta_{\bar{u}}}{\beta}\right)+\frac{1}{16}\frac{\beta_{\bar{u}}^2}{\beta^2}\right], \\
k &= \alpha'^2\left[g+\alpha'_{\bar{u}}-\frac{1}{4}\frac{\partial}{\partial\bar{v}}\left(\frac{\alpha'_{\bar{v}}}{\alpha'}\right)+\frac{1}{16}\frac{\alpha'^2_{\bar{v}}}{\alpha'^2}\right]
\end{aligned}
\tag{19}
$$

are the invariants sought. In calculating their expressions in terms of the coefficients, we must note that in (19) occur differentiations with respect to the transformed variables $\bar{u}$ and $\bar{v}$. The relation between such differentiations and those with respect to the original variables is given by the formulæ (4),

in which y is to be interpreted as any function of u, v. Solving for $\bar{y}_u$ and $\bar{y}_v$, we have

$$(20)\quad \bar{y}_u = -\frac{1}{\phi_u}\left(\frac{1-\mu}{2\mu}y_u - \frac{a}{2\mu}y_v\right), \qquad \bar{y}_v = \frac{1}{\psi_v}\left(\frac{a'}{2\mu}y_u - \frac{1-\mu}{2\mu}y_v\right).$$

Consequently,

$$\alpha_{\bar{u}} = -\frac{1}{\phi_u}\left[\frac{1-\mu}{2\mu}\frac{\{\alpha\}_u + \tfrac{1}{2}\xi_u - \{\alpha\}\xi - \tfrac{1}{2}\xi^2}{\phi_u} - \frac{a}{2\mu}\frac{\{\alpha\}_v + \tfrac{1}{2}\xi_v - \dfrac{\phi_{uv}}{\phi_u}(\{\alpha\} + \tfrac{1}{2}\xi)}{\phi_u}\right].$$

But ξ_v in this expression may be found in terms of ξ_u and ξ. In fact, we have

$$(21)\quad \begin{aligned}\xi_v &= \frac{\partial}{\partial v}\left(\frac{\phi_{uu}}{\phi_u}\right) = \frac{\phi_{uuv} - \xi\phi_{uv}}{\phi_u}\\ &= \left(\frac{1+\mu}{a}\right)_{uu} + \xi\left(\frac{1+\mu}{a}\right)_u + \frac{1+\mu}{a}\xi_u,\end{aligned}$$

by the first of equations (12). Therefore

$$\alpha_{\bar{u}} = -\frac{1}{\phi_u^2}\left[\frac{1-\mu}{2\mu}\{\alpha\}_u - \frac{a}{2\mu}\left[\{\alpha\}_v - \{\alpha\}\left(\frac{1+\mu}{a}\right)_u + \tfrac{1}{2}\left(\frac{1+\mu}{a}\right)_{uu}\right] - \tfrac{1}{2}\xi_u + \{\alpha\}\xi + \tfrac{1}{2}\xi^2\right],$$

so that

$$(22)\qquad f = \frac{1}{\phi_u^2}(\{f\} - \tfrac{1}{2}\xi_u + \tfrac{1}{4}\xi^2 - \{\beta\}\eta),$$

where

$$(23)\quad \begin{aligned}\{f\} = \{\gamma\} - 2\{\beta\}\{\beta'\} - \{\alpha\}^2 + \frac{1-\mu}{2\mu}\{\alpha\}_u\\ - \frac{a}{2\mu}\left[\{\alpha\}_v - \{\alpha\}\left(\frac{1+\mu}{a}\right)_u + \tfrac{1}{2}\left(\frac{1+\mu}{a}\right)_{uu}\right].\end{aligned}$$

It may also be found without difficulty that

$$(24)\qquad \beta_{\bar{u}} = -\frac{\psi_v}{\phi_u^3}(\{\beta_{\bar{u}}\} + 2\{\beta\}\xi), \qquad \beta_{\bar{v}} = \frac{1}{\phi_u^2}(\{\beta_{\bar{v}}\} + \{\beta\}\eta),$$

where the quantities

$$(25)\quad \begin{aligned}\{\beta_{\bar{u}}\} &= \left[\frac{a}{\mu}\left(\frac{1+\mu}{a}\right)_u + \frac{1-\mu}{2\mu}\left(\frac{1+\mu}{a'}\right)_v\right]\{\beta\} + \frac{1-\mu}{2\mu}\{\beta\}_u - \frac{a}{2\mu}\{\beta\}_v,\\ \{\beta_{\bar{v}}\} &= \left[\frac{1-\mu}{\mu}\left(\frac{1+\mu}{a}\right)_u + \frac{a'}{2\mu}\left(\frac{1+\mu}{a'}\right)_v\right]\{\beta\} + \frac{a'}{2\mu}\{\beta\}_u - \frac{1-\mu}{2\mu}\{\beta\}_v\end{aligned}$$

are the parts of $\beta_{\bar{u}}$ and $\beta_{\bar{v}}$ independent of ϕ and ψ. Continuing to use braces $\{\}$ to denote that part of the enclosed expression which does not contain ϕ or ψ, we have

$$\frac{\partial}{\partial \bar{u}} \frac{\beta_{\bar{u}}}{\beta} = \frac{1}{\phi_u^2}\left(\left\{ \frac{\partial}{\partial \bar{u}} \frac{\beta_u}{\beta} \right\} - 2\xi_u + \frac{\{\beta_{\bar{u}}\}}{\{\beta\}} \xi + 2\xi^2\right), \tag{26}$$

where

$$\left\{ \frac{\partial}{\partial \bar{u}} \frac{\beta_{\bar{u}}}{\beta} \right\} = \frac{1-\mu}{2\mu} \frac{\partial}{\partial u} \frac{\{\beta_{\bar{u}}\}}{\{\beta\}} - \frac{a}{2\mu}\left[\frac{\partial}{\partial v} \frac{\{\beta_{\bar{u}}\}}{\{\beta\}} - \left(\frac{1+\mu}{a}\right)_u + 2\left(\frac{1+\mu}{a}\right)_{uu} \right]. \tag{27}$$

Using (15), (22), (24), and (26), the first of the invariants (19) becomes

$$k = \frac{\psi_v^2}{\phi_u^6}\{h\}, \tag{28}$$

where

$$\{h\} = \{\beta\}^2\left[\{f\} + \{\beta_{\bar{v}}\} - \frac{1}{4}\left\{ \frac{\partial}{\partial \bar{u}} \frac{\beta_{\bar{u}}}{\beta} \right\} + \frac{1}{16} \frac{\{\beta_{\bar{u}}\}^2}{\{\beta\}^2} \right]. \tag{29}$$

Similarly, we find for the invariant k

$$k = \frac{\phi_u^2}{\psi_v^2}\{k\}, \tag{30}$$

where the formula for $\{h\}$ may be used to get the expression for $\{k\}$, merely by replacing any letter in the symbol

$$\begin{pmatrix} \alpha & \beta & \gamma & a & u & \phi & \xi & f \\ \beta' & \alpha' & \gamma' & a' & v & \psi & \eta & g \end{pmatrix}$$

by the letter just above or just below it.

From the four invariants β, α', h, k, that is,

$$\beta = \frac{\psi_v}{\phi_u^2}\{\beta\}, \quad \alpha' = \frac{\phi_v}{\psi_v^2}\{\alpha'\}, \quad h = \frac{\psi_v^2}{\phi_u^6}\{h\}, \quad k = \frac{\phi_u^2}{\psi_v^6}\{k\}, \tag{31}$$

a complete system of invariants may be calculated. Let us put, with Wilczynski,*

$$A = \alpha' \beta^2, \quad B = \alpha'^2 \beta, \quad H = \alpha' h, \quad K = \beta k; \tag{32}$$

then

$$A = \frac{1}{\phi_u^3}\{A\}, \quad B = \frac{1}{\psi_v^3}\{B\}, \quad H = \frac{1}{\phi_u^5}\{H\}, \quad K = \frac{1}{\psi_v^5}\{K\}, \tag{33}$$

where $\{A\}$, $\{B\}$, $\{H\}$, and $\{K\}$ again denote functions independent of ϕ and ψ. Wilczynski has proved† that all invariants of the system (14) may be calculated from A, B, H, and K by combination and repetition of certain differ-

* *First Memoir*, p. 250.

† *First Memoir*, pp. 250–255.

entiation processes. First, we note that if system (14) be transformed by

$$\bar{u} = U(\bar{u}), \qquad \bar{v} = V(\bar{v}),$$

a new system (which we may call $(\overline{14})$), of the same form is obtained, the invariants $\bar{A}, \bar{B}, \bar{H}, \bar{K}$ of which are related to the invariants A, B, H, K of (14) by the equations*

$$(34) \qquad \bar{A} = \frac{1}{U_{\bar{u}}^3} A, \qquad \bar{B} = \frac{1}{V_{\bar{v}}^3} B, \qquad \bar{H} = \frac{1}{U_{\bar{u}}^5} H, \qquad \bar{K} = \frac{1}{V_{\bar{v}}^5} K.$$

We shall call any invariants L and M of system (14) which are connected with the corresponding quantities for the transformed system $(\overline{14})$ by equations of the form

$$(35) \qquad \bar{L} = \frac{1}{U_{\bar{u}}^l} L, \qquad \bar{M} = \frac{1}{V_{\bar{v}}^m} M,$$

U- and V-invariants respectively.

Then the operator

$$(36) \qquad \mathbf{U} = \alpha' \frac{\partial}{\partial \bar{u}},$$

applied to a V-invariant, gives another V-invariant, and the operator

$$(37) \qquad \mathbf{V} = \beta \frac{\partial}{\partial \bar{v}},$$

applied to a U-invariant, gives another U-invariant. In fact, we find

$$(38) \qquad \bar{\mathbf{V}}(\bar{L}) = \frac{1}{U_{\bar{u}}^{l+2}} \mathbf{V}(L), \qquad \bar{\mathbf{U}}(\bar{M}) = \frac{1}{V_{\bar{v}}^{m+2}} \mathbf{U}(M).$$

Again, from two U-invariants P and Q which satisfy the relations

$$(39) \qquad \bar{P} = \frac{1}{U_{\bar{u}}^p} P, \qquad \bar{Q} = \frac{1}{V_{\bar{v}}^q} Q,$$

may be formed their wronskian with respect to $\bar{u}$, viz.,

$$(40) \qquad (QP_{\bar{u}}) = qQP_{\bar{u}} - pPQ_{\bar{u}},$$

an invariant satisfying the equation

$$(41) \qquad (\overline{QP}_{\bar{u}}) = \frac{1}{U_{\bar{u}}^{p+q+1}} (QP_u).$$

Similarly, from two V-invariants may be formed their wronskian with respect to $\bar{v}$.

* *First Memoir*, pp. 249, 250. Our equations (34) are Wilczynski's equations (52), with α, β, u, v replaced by $U, V, \bar{u}, \bar{v}$.

Wilczynski's theorem is, that *all invariants of system* (14) *may be calculated from the fundamental ones, A, B, H, K, by combination and repetition of the wronskian-, U-, and V-processes.* It remains for us to show that *the invariants thus found are expressible explicitly in terms of the coefficients and variables of the original system of differential equations* (2).

The proof is a very simple induction. Assume that invariants L and M of system (14) which satisfy (35) have the following expressions in terms of the coefficients and variables of system (2):

$$L = \frac{1}{\phi_u^l}\{L\}, \qquad M = \frac{1}{\psi_v^m}\{M\}. \tag{42}$$

This assumption is verified by the invariants A, B, H, K, as is evident on comparing equations (33) and (34). From equations (20) we have

$$\begin{aligned} \mathbf{U} &= \frac{1}{\psi_v^2}\{\alpha'\}\left(\frac{a}{2\mu}\frac{\partial}{\partial v} - \frac{1-\mu}{2\mu}\frac{\partial}{\partial u}\right), \\ \mathbf{V} &= \frac{1}{\phi_u^2}\{\beta\}\left(\frac{a'}{2\mu}\frac{\partial}{\partial u} - \frac{1-\mu}{2\mu}\frac{\partial}{\partial v}\right), \end{aligned} \tag{43}$$

and it is easy to verify, if use be made of equations (12), that

$$\mathbf{U}(M) = \frac{1}{\psi_v^{m+2}}\{\mathbf{U}(M)\}, \qquad \mathbf{V}(L) = \frac{1}{\phi_u^{l+2}}\{\mathbf{V}(L)\}, \tag{44}$$

where

$$\begin{aligned} \{\mathbf{U}(M)\} &= \{\alpha'\}\left[\frac{a}{2\mu}\{M\}_v - \frac{1-\mu}{2\mu}\{M\}_u + \frac{1-\mu}{2\mu}\left(\frac{1+\mu}{a'}\right)_v\{M\}\right], \\ \{\mathbf{V}(L)\} &= \{\beta\}\left[\frac{a'}{2\mu}\{L\}_u - \frac{1-\mu}{2\mu}\{L\}_v + \frac{1-\mu}{2\mu}\left(\frac{1+\mu}{a}\right)_u\{L\}\right]. \end{aligned} \tag{45}$$

Similarly for the wronskian, it may be shown that if

$$P = \frac{1}{\phi_u^p}\{P\}, \qquad Q = \frac{1}{\phi_u^q}\{Q\}, \tag{46}$$

then

$$(QP_{\bar{u}}) = \frac{1}{\phi_u^{p+q+1}}\{(QP_{\bar{u}})\}, \tag{47}$$

where

$$\begin{aligned} \{(QP_{\bar{u}})\} = q\{Q\}\left(\frac{a}{2\mu}\{P\}_v - \frac{1-\mu}{2\mu}\{P\}_u\right) \\ - p\{P\}\left(\frac{a}{2\mu}\{Q\}_v - \frac{1-\mu}{2\mu}\{Q\}_u\right). \end{aligned} \tag{48}$$

Now, equations (44) and (47) resemble equations (38) and (41) just as equations (42) and (46) resemble by assumption equations (35) and (39). But

this assumed resemblance exists for the fundamental invariants A, B, H, K; consequently it will be preserved for all invariants formed from these by the wronskian-, U-, and V-processes. By this method, however, all invariants of system (14) may be formed,* so that the complete system of invariants so constructed is expressible explicitly in terms of the coefficients of system (2), except of course for extraneous factors of the form $\phi_u^s \psi_v^t$ in the case of relative invariants.

In the same way, the four fundamental covariants of Wilczynski's theory may be expressed in terms of the coefficients and variables of system (2). We take as these covariants†

$$y, \qquad Z = 6Az + A_u y, \qquad R = 6B\rho + B_v y,$$

$$S = 36AB\sigma + 6AB_v z + 6BA_u \rho + A_u B_v y,$$

where

$$z = y_u + \alpha y, \qquad \rho = y_v + \beta' y,$$

$$\sigma = y_{uv} + \beta' y_u + \alpha y_v + \tfrac{1}{2}(\alpha_v + \beta'_u + 2\alpha\beta') y.$$

It may be verified without difficulty that

$$Z = \frac{1}{\phi_u^4}\{Z\}, \qquad R = \frac{1}{\psi_v^4}\{R\}, \qquad S = \frac{1}{\phi_u^4 \psi_v^4}\{S\},$$

where the quantities $\{Z\}$, $\{R\}$, $\{S\}$ are easily calculated expressions in the coefficients and variables of system (2), and do not contain either ϕ or ψ.

It is therefore possible to express all of the concomitants, as calculated by Wilczynski for system (14), in terms of the coefficients and variables of system (2), even though the reduction of (2) to (14) requires the integration of the partial differential equations (8). That this must be so becomes sufficiently obvious if one inquires into the geometric significance of the invariants and covariants. As calculated for system (14), these invariants and covariants have interpretations connected with the asymptotic lines. But the properties of the asymptotic lines so expressed are such that these properties may be recognized, not only if the asymptotic lines themselves are known, but even if merely the two asymptotic directions at each point of the surface are known. These asymptotic directions, however, may be determined without any integration, so that no integration ought to be necessary in expressing the properties of the surface, as defined by (2), in terms of the asymptotic lines.

* Wilczynski, *First Memoir*, p. 255.

† The quantities z, ρ, σ are the semi-covariants given by equations (40) of the *First Memoir*. Of the quantities (69) of that memoir, the first three are covariants; Professor Wilczynski has remarked to the writer that the fourth of (69) is not a covariant, and suggests the substitution of the quantity S instead. It should be remembered that the Greek letters α, β, γ replace the Roman a, b, c of the *First Memoir*.

Our analytic work proves that this is in fact the case. As an example, take the condition that the surface be ruled. In terms of the asymptotic lines, this is the condition that one of the families of asymptotic lines be composed of straight lines. Now, given the ∞^2 asymptotic directions corresponding to a family of asymptotic curves, it is evident that no integration is required to determine whether these ∞^2 directions give ∞^1 straight lines. Whether a similar statement may be made for *any* property of the surface which may be expressed in terms of the concomitants is not of course *a priori* obvious; our analytic results, however, actually prove that this is so. *All properties of the surface expressible in terms of the invariants and covariants of equations* (2) *are such that corresponding properties of the asymptotic lines of the surface may be enunciated in terms of the same invariants and covariants, even though it be impossible to determine the asymptotic lines themselves.*

A discussion analogous to that of the present paper finds a place in the study of many configurations by means of Wilczynski's general method. According to this method, a projective theory is equivalent to the theory of the invariants and covariants of a completely integrable system of differential equations, say S. It is, however, frequently convenient to use instead of S a canonical form, say S', on account of analytic simplifications introduced thereby. But to throw S into the canonical form S' may require an impossible integration. Nevertheless, the invariants and covariants calculated for the system S' may be calculable explicitly in terms of the coefficients and variables of the original system S. In that case—already exemplified by the calculations of the present paper—it is possible to make use of the simplifications, due to the canonical form, in the actual calculation of the concomitants for the general system S.

The above remarks, however, find their chief importance in the obvious deduction that, without any loss of generality whatever, the canonical system S' may be used instead of the original system S. The restriction of the problem consists merely in the omission of calculations which it is always possible to carry out in practice.

COLLEGE OF THE CITY OF NEW YORK,
January, 1914.

THE MULTITUDE OF TRIAD SYSTEMS ON 31 LETTERS*

BY

HENRY S. WHITE

Triad systems, or triple systems, apparently different, on 15 letters have long been known; and as early as 1859 Reiss established the existence of at least one triad system for every number of letters, t, of the form $6k + 1$ or $6k + 3$. E. H. Moore in 1893 was the first to prove† that when t is above 13 there are at least two non-equivalent systems. This he proved by showing a difference in the groups that leave the two systems respectively invariant. He found reason also for the conjecture that a particular class of such systems, those whose group contains a cyclic substitution of order t, increase in number rapidly with increasing t. For the number $t = 13$, which Moore had left in doubt, Zulauf then proved* that there are at least two distinct systems; and it was demonstrated finally by De Pasquale† (later also by Brunel‡) that for $t = 13$ there are only those two systems. Zulauf's proof was by the use of the groups of the systems.

While two systems belonging to different groups are obviously not equivalent—that is, they cannot be transformed into one another by any substitution on the letters, Miss Cummings§ has found that the converse is not true, and has given examples of two systems having the same group, which yet are not equivalent. She has applied a new method of comparison, that of sequences and indices, and has constructed for $t = 15$ so many new systems that their number is now known to be not less than 24. I have now been able, by a reasonably short method, to make for $t = 15$ an exhaustive catalogue of all triad systems whose group is of order at least 2. By this means there are disclosed a considerable number additional, of a special kind previously

* Presented to the Society, September 8, 1914.

† E. H. Moore, *Concerning triple systems*, Mathematische Annalen, vol. 43 (1893), pp. 271–285.

* K. Zulauf, *Ueber Tripelsysteme von 13 Elementen*. Dissertation, Marburg, 1897.

† *Sui sistemi ternari di 13 elementi*. Rendiconti R. Istituto Lombardo, ser. 2, vol. 32 (1899).

‡ *Sur les deux systèmes de triades de treize éléments*. Journal de Mathématiques, ser. 5, vol. 7 (1901). See also an article cited later, by F. N. Cole, containing a complete demonstration.

§ *On a method of comparison for triple-systems*. These Transactions, vol. 15 (1914), p. 311.

known only in a single example. The method and results of this study will soon be laid before this Society; but my present purpose is to apply some of the results to the next question in order.

As soon as we possess one or more triad systems on 15 letters, systems on 31 can be constructed. With the aid of a new theorem, and a novel property of the arrangement of the 120 pairs on 15 letters in an "array" of 15 columns of 8, I succeed in constructing a simple kind of system on 31 letters and proving that a very great number of such systems are mutually non-equivalent. How many kinds there are, and even the precise number of this most simple kind, I do not determine. But we do see that their number is so overwhelmingly great that further study cannot be that of details and individual systems, but must necessarily be a search for general characters, relations, and classification.

From the small number of triad systems hitherto written out explicitly, and the slow rate of their increase up to $t = 15$, the totality of triad-systems has been considered to be comparable to that of polyedra. From present indications, however, their field appears to be far more extensive. Its fertility is yet to be determined.

1. The odd and even structure

Certain triad systems admit a separation of their letters into two sets, those of the one set appearing by ones or threes in all the triads, while those of the other set appear (consequently) either two together or not at all. Those sets, when the separation is possible, we term respectively an odd set and an even set. It is easily seen that the even set must contain one more letter than the odd set; and that the triads containing the odd set exclusively must constitute a closed system, a complete triad system contained in the larger one. Every triad system on 15 letters that has been described up to the present time exhibits this odd and even structure, i. e., contains a triad system on 7 letters (a Δ_7) and an even set of 8; with one exception, namely the cyclic Δ_{15} of Heffter. We shall call the odd set the *head*, and so classify Heffter's cyclic system as *headless*. Using the usual notation we say that a Δ_{2n+1} exhibiting the odd and even structure has for head a Δ_n. If, conversely, a Δ_{2n+1} contains a Δ_n, it has odd and even structure with that Δ_n for odd set. Often a Δ_{2n+1} is found to contain more than one Δ_n, and so to be separable in more than one way into two sets.

Theorem. *If a Δ_{15} contains two Δ_7's, they have in common one triad and no other letters. If a Δ_{31} contains two Δ_{15}'s, they have in common seven triads constituting a Δ_7, but no other letters.*

Proof.—It is sufficient to demonstrate the second statement, for the same proof applies to the general theorem. Let the Δ_{31} contain Δ_{15} and Δ'_{15}, two triad systems on 15's. Denote the set of 15 letters in Δ_{15} by $a, b, c, \cdots$, and

its even set of 16 by $\alpha, \beta, \gamma, \cdots, \pi$. The Δ'_{15}, as different from Δ_{15}, must contain letters from both sets. Call one even letter in Δ'_{15} π. This letter π will occur in 7 triads of Δ'_{15}, and since π is of the even set in Δ_{31}, these triads are of the type $\pi a\alpha$. Consequently the Δ'_{15} contains, beside π, seven even letters and seven odd. Any triad containing 2 of these 7 odd letters must contain a third odd letter; they constitute therefore a closed set of 7 triads, a Δ_7, both in Δ_{15} and in Δ'_{15}. Further we note that the 8 even letters, $\pi, \alpha, \cdots$, occurring in Δ'_{15} are excluded from Δ_{15}; and the theorem is proved.

There are 8 residual letters of the even set, excluded both from Δ_{15} and Δ'_{15}. These must form, with each of the 7 odd letters common to both systems, 4 triads of the type $a\alpha\beta$; in all, 28 triads. Combine the 7 common triads with these 28 external to both systems, and we see a third system, Δ''_{15}, containing the 7 letters of the common set and the residual 8 even letters. We therefore state the theorem more fully thus:

THEOREM. *If a Δ_{31} contains two Δ_{15}'s, it contains also a third; and the three Δ_{15}'s have in common a Δ_7, but no letters except those involved in the Δ_7. The three together involve all 31 letters, and contain 91 of the 155 triads that occur in the Δ_{31}.*

COROLLARY 1. *If a Δ_{31} contains a Δ_{15} which contains no Δ_7 (a headless Δ_{15}), then the Δ_{31} contains no other Δ_{15}.*

COROLLARY 2. *If a Δ_{31} contains a Δ_{15} which contains only one Δ_7, then it cannot contain more than two other Δ_{15}'s.*

COROLLARY 3. *A Δ_{31} which contains a Δ_{15} of the kind showing only 3 Δ_7's cannot have more than seven Δ_{15}'s in all.* The remaining possibilities are evidently fifteen and thirty-one.

2. SYNTHESIS OF A Δ_{31} OF ODD AND EVEN STRUCTURE

It is the first of the preceding corollaries that I wish to employ here in proving non-equivalence. There is a well-known method, generalized from Reiss, of constructing a Δ_{31} by the use of two Δ_{15}'s. These may be either different or equivalent systems; the one is used in forming an array, 15 columns containing 8 pairs of letters each, the other is then superposed, in the form of 15 columns, each of which consists of 7 pairs with one letter at the head of the column. The letters of this latter system, $a, b, c, \cdots$, are set as column-heads along a line, and in column below each one are written the 7 pairs or dyads that occur with it in triads of the system. To form the 15×8 array, the first system, consisting of 15 other letters $\alpha, \beta, \gamma, \cdots$, is written out in a similar tabular form, then with each letter at the head of a column is written a 16th letter, as x. This gives us all the 120 pairs that can be made from the 16 letters $\alpha, \beta, \gamma, \cdots, x$, with no pair repeated. These 15 columns of 8, set below the other 15 columns of 7 in any order, will constitute a table of a

Δ_{31}, — the odd set of 15 above, the even set of 16 below. *If the first* Δ_{15}, *used as here described, is of the headless variety* (*contains no* Δ_7), *the resulting* Δ_{31} *will contain no other* Δ_{15}. For we have then the case of Corollary 1, § 1.

As stated already, the order of columns in the lower array is entirely arbitrary, so that the number of possible juxtapositions when the upper system is superposed will be 15! But the number of non-equivalent Δ_{31}'s resulting may be less. Any substitution on the letters $a, b, c, \cdots$ that transforms the superposed Δ_{15} into itself will merely shift the relative positions of its columns; and similarly any substitution belonging to the group of the lower array, the Δ_{31} being transformed in either case into an equivalent Δ_{31} (by the definition of equivalence). Our proposed permutations of entire columns in the lower array constitute a group $G_{15!}$. If two resulting systems are equivalent, it must be through the equivalence of their unique Δ_{15}'s in the upper array and the equivalence of their respective lower arrays. But the upper arrays are identical, whence their equivalence must subsist in a transformation of that upper array (i. e., of the headless Δ_{15} which it tabulates) into itself. The group of that headless Δ_{15} is assumed as given, so that we have now to inquire what subgroup of the $G_{15!}$ on the lower columns is capable of being induced by substitutions on the 16 letters of that array? The problem of equivalence is now resolved into two problems of invariance, one solved in advance, the other more intricate, as will be seen.

3. The group of a 15 by 8 array derived from a Δ_{15}

The even set of a Δ_{31}, when the system is separable into even and odd sets, can be displayed in fifteen columns, each containing eight pairs formed from letters of this even set, while above each column stands the letter of the odd set which completes its pairs to triads. Such an array is not always derivable from a Δ_{15} by the method explained in section 2 above, and we do not propose here the general problem of its possible varieties. We require for present purposes only those that are so derivable from some triad system on 15 letters. Let the system contain a triad $\alpha\beta\gamma$. This will give rise to pairs in three columns, those headed by α, β, and γ respectively. After a letter x has been combined with each such head, the three columns will contain two pairs each, related like the following:

$$\begin{array}{ccc} x\alpha & x\beta & x\gamma \\ \beta\gamma & \alpha\gamma & \alpha\beta. \end{array}$$

These exhibit what Cole terms triple interlacing,* and Miss Cummings† styles

* F. N. Cole, *The triad systems of 13 letters*. These Transactions, vol. 14 (1913), p. 1.

† L. D. Cummings. These Transactions, vol. 15 (1914), p. 311.

a complete quadrangle. As there are 35 triads in a Δ_{15}, the particular letter x will participate in 35 such quadrangles; each pair $x\alpha$ with all seven other pairs in its column will give rise to seven quadrangles. $((15 \times 7) \div 3 = 35.)$ Conversely, of course, if in any such 15 by 8 array there is one letter x that occurs in 35 quadrangles, that array is derived from a Δ_{15}, namely from the Δ_{15} whose tabulation appears upon the removal of that letter x from the pair at the head of each column. This observation provides a test easily applied; for if two pairs in any column are seen not to be concerned in a complete quadrangle, none of the four letters involved can be the letter x of this discussion.

This test will reveal of course all the different Δ_{15}'s that may be sources of any given 15 by 8 array. We apply it to arrays that have been formed from all Δ_{15}'s now known (including headless Δ_{15}'s not yet published), in order to learn whether any such array can have two different Δ_{15}'s for its source. The result is negative in every case, with two exceptions, namely the systems III A and III B of Miss Cummings. *Any* 15 *by* 8 *array containing the* 120 *pairs that can be formed from* 15 *letters, no letter occurring twice in a column, if it is derivable from more than one* Δ_{15}, *must come from either the Kirkman system, III A, or from the system III B of Miss Cummings's dissertation.*

These two exceptional cases we shall exclude from present consideration, and so simplify the question of the group of an array. Any substitution on its 16 letters which merely shifts the 15 columns must evidently also transform into itself any Δ_{15} which can be a source, or else interchange among themselves two or more such sources. As we shall consider only those arrays that have a unique Δ_{15} as source, we adopt always the first of these alternatives. *The group of the array is the same as that of the* Δ_{15} *from which it is derived.* Or better, if we think of the group of an array as operating directly, not on its letters, but upon the numbers that denote the order of succession of its columns, we should say that the two groups are holoedrically isomorphic. But our question was this: How many permutations of the entire columns of the array are the same in effect as a mere interchange among the 16 letters? This question too has been answered by implication when we answered its converse.

Let us recapitulate, since this is the essential point in our proof. We have a superposed Δ_{15}, the choice of which insures its uniqueness in the Δ_{31}, arranged in 15 columns. Below these we have the 15 columns of an array, 15 by 8, derivable from only one Δ_{15}. The columns of this array are permuted in 15! ways; and it is found that equivalent Δ_{31}'s are produced only when one permutation is convertible into the other either by (1) an operation on the superposed partial columns belonging in the group of their Δ_{15}, or

by (2) an operation in the group of the Δ_{15} from which the lower array of partial columns is derived, or by a combination of both. The orders of these two groups being denoted by d and d' respectively, we now remark further that the number of operations common to the two groups (call it r), will be the order of the group of the resulting Δ_{31}, and that the product dd' divided by r will give the number of systems Δ_{31} equivalent to any one. Of course r need not be the same for two non-equivalent sets of arrangements, but no such set of equivalent arrangements can number more than dd'. For forming an approximate idea, this number dd' will be used, although it is often too large. *Then for any one superposed* Δ_{15} *and any one subterposed array, the number of non-equivalent* Δ_{31}*'s produced cannot be less than* $15!/(dd')$.

4. Compound triad systems on 31 letters, whose odd set constitute a headless Δ_{15}

It is the intention to prove that the Δ_{31}'s are exceedingly numerous, not to find their number with any precision. It will suffice if we examine roughly triad systems, Δ_{31}'s produced as above described, which are compounded in sets of not more than 15 equivalent systems. This is an arbitrary bound, and we shall further reckon as if 15 were the precise number of conjugate, equivalent systems in each set. I take from Miss Cummings's dissertation (loc. cit.), and from my own unpublished work* already referred to, data upon Δ_{15}'s whose groups are of low order. A headless Δ_{15} whose group is of order d will be used as head (i. e., odd set) for a Δ_{31}; and to furnish the even set, the 15 by 8 array, any Δ_{15}, either the same or different, headless or itself compound, will be employed. We shall avoid, for the reasons stated. the Cummings systems III A and III B, so that never will the same array come from two sources and never will equivalent Δ_{31}'s be counted as distinct. The table below is arranged for values of d and d', showing in how many distinct Δ_{15}'s each occurs. Of course then the d' relates to both classes of Δ_{15}'s, the d to headless systems only.

Values of d	Values of d'	Resulting divisors, dd', of 15!
2, 4 systems	2, 7 systems	4, 28 combinations
3, 8 "	3, 12 "	6, 104 "
4, 4 "	4, 7 "	8, 56 "
5, 1 system	5, 1 system	9, 96 "
6, 1 "	6, 1 "	10, 11 "
	8, 2 systems	12, 115 "
		15, 20 "
		Total, 430 divisors not above 15.

* This research has later been communicated to the American Mathematical Society, October 31, 1914.

If these were all, and if these all arose in sets of 15 conjugate equivalent systems, even then their number would be 430 times 15!/15, i. e. 430 (14!), which is above 3×10^{13}.

Certainly therefore there exist more than thirty-seven million millions of triad systems on 31 *letters, no two systems equivalent, of that restricted kind which contain one* Δ_{15}, *which itself contains no* Δ_7.

This leaves untouched many questions as to other kind of arrays and other kinds of Δ_{31}'s, and particularly any kind that may contain no Δ_{15}.

Similar reasoning carried out for triad systems on 27 letters, constructed from the two kinds on 13 letters whose groups are of orders 6 and 39 respectively, shows a number greater than 222 millions.

Let me call attention also to the fact that Δ_{31}'s actually exist whose group is the identity only. For if d and d' are relatively prime, the subgroup which forms the group of the Δ_{31} must be of order 1. This kind of triad systems is found therefore in great number for 31 letters, but none have yet been announced for a less number of letters.

VASSAR COLLEGE,
August, 1914.

THE ϕ-SUBGROUP OF A GROUP*

BY

G. A. MILLER

1. INTRODUCTION

Any set of operators belonging to any group G of finite order is called a set of independent generators of G provided that these operators generate G and that none of them is contained in the group generated by the rest of them. All the operators of G can be divided into two categories, having no common operator, by putting into one category all those which occur in at least one of the possible sets of independent generators of G, and into the other category those which do not have this property. The operators of the second category constitute a characteristic subgroup of G, which has been called† by G. Frattini the *ϕ-subgroup of G*.

If H is any maximal subgroup of G it is evidently always possible to select at least one set of independent generators of G in such a manner that it includes any arbitrary one of the operators of G which are not contained in H while the remaining operators of the set belong to H. Moreover, there is at least one maximal subgroup of G which does not include any given one of the independent generators of a particular set of independent generators of G. Hence it results that the ϕ-subgroup of G is the cross-cut of all the maximal subgroups of G. This useful second definition of the ϕ-subgroup is also due to Frattini.

As every maximal subgroup of a group of order p^m, p being a prime number, is of order p^{m-1}, it results directly from this second definition that *the ϕ-subgroup of any group of order p^m is the cross-cut of all its subgroups of index p*. It results from this that the ϕ-subgroup of a group of order p^m may also be defined as its smallest invariant subgroup which gives rise to an abelian quotient group of type $(1, 1, 1, \cdots)$.‡ If the order of this quotient group is p^α, it results that α is the number of independent generators in every possible set of independent generators of this group of order p^m. In particular, *if the*

* Presented to the Society, September 8, 1914.

† G. Frattini, Atti della Reale Accademia dei Lincei, Rendiconti, ser. 4, vol. 1 (1885), p. 281.

‡ M. Bauer, Nouvelles Annales de Mathématiques, ser. 3, vol. 19 (1900), p. 509.

order of G is p^m, p being any prime number, every possible set of independent generators of G involves the same number of operators. That is, the number of operators in each of the possible sets of independent generators of any Sylow group is an *invariant* of the group.

From the preceding paragraph it results directly that a necessary and sufficient condition that the ϕ-subgroup of a given group of order p^m be the identity is that this group be the abelian group of type $(1, 1, 1, \cdots)$. Hence there is one and only one group of order p^m, p being any prime number and m being any positive integer, which has the identity for its ϕ-subgroup. The number of operators in every set of independent generators of this group is m. In every other group of order p^m the number of these independent generators is less than m, and there is at least one group of order p^m in which this number is any arbitrary positive integer from 1 to m.

If a ϕ-subgroup of the group G involves a non-invariant subgroup or a non-invariant operator this subgroup or operator cannot be transformed into all its conjugates under G by the operators of the ϕ-subgroup. That is, every complete set of conjugates of the ϕ-subgroup is an incomplete set of conjugates under G whenever the former set involves more than one element. If this were not the case all the operators of G which would transform one of these elements into itself would form a subgroup which would not involve all the operators of the ϕ-subgroup of G. This subgroup could not be maximal since it does not involve the ϕ-subgroup. As any maximal subgroup obtained by extending this subgroup by means of operators of G could also not involve the ϕ-subgroup we have proved the theorem: *If the ϕ-subgroup of a group G involves a non-invariant operator or subgroup, the number of conjugates under G of this operator or subgroup is greater than the number of the corresponding conjugates under the ϕ-subgroup.*

An important special case of this theorem was noted by Frattini who observed that the ϕ-subgroup of any group involves only one Sylow subgroup for every prime which divides the order of the ϕ-subgroup. In other words, every ϕ-subgroup is the direct product of its Sylow subgroups, and hence we can always reach the identity by forming successive ϕ-subgroups, starting with any given group. If a group can be represented as a non-regular primitive substitution group of degree n, its n subgroups of degree $n - 1$, composed of all its substitutions which omit a letter, are maximal and have only the identity in common. Hence it results that *the ϕ-subgroup of every primitive substitution group is the identity.*

While the number of the possible independent generators of a group whose order is a power of a prime is an invariant of the group, this number is not always an invariant as regards other groups. For instance, it is evident that if we use transpositions as independent generators of the symmetric group

of degree n the number of these generators is always $n-1$. On the other hand, this number is always two if we use a cyclic substitution of degree $n-1$ and a transposition involving the remaining letter as the independent generators of this symmetric group. Hence it results that it is possible to select a set of independent generators of the symmetric group of degree n so that the number of operators in the set is an arbitrary integer from 2 to $n-1$.

2. The ϕ-subgroup of an abelian group

If G is an abelian group of order p^m, p being any prime number, the ϕ-subgroup of G is the subgroup composed of the distinct operators obtained by raising every operator of G to its pth power. As a *reduced set of independent generators** of this ϕ-subgroup we may therefore take the pth powers of each operator whose order exceeds p in any reduced set of independent generators of G. The order of the ϕ-subgroup of G is therefore $p^{m-\alpha}$, α being the number of the invariants of G. In other words, p^α is the order of the subgroup of G generated by all of its operators of order p.

If any group (abelian or non-abelian) is the direct product of two groups it results that the direct product of one of these two factor groups and a maximal subgroup of the other is a maximal subgroup of the original group. Since the ϕ-subgroup of any group is both invariant and occurs in every maximal subgroup it results, as was observed by G. Frattini, that *every maximal invariant subgroup of any group (abelian or non-abelian) involves the ϕ-subgroup of this group.* By combining these two results we have a proof of the following useful theorem.

THEOREM. *The ϕ-subgroup of the direct product of any two groups (abelian or non-abelian) is the direct product of the ϕ-subgroups of these two groups.*

As every abelian group whose order is not a power of a prime number is the direct product of its Sylow subgroups it results directly from the preceding theorem that the ϕ-subgroup of any abelian group is the direct product of the ϕ-subgroups of its Sylow subgroups. A necessary and sufficient condition that the ϕ-subgroup of an abelian group be the identity is that each of its Sylow subgroups be of type $(1, 1, 1, \cdots)$. There is therefore one and only one abelian group of every possible order which has the identity for its ϕ-subgroup.

If $s_1, s_2, \cdots, s_\lambda$ is any set of independent generators of an abelian group G of order p^m, it is easy to derive a *reduced* set of λ independent generators from the given set. In fact, any operator of highest order in this set may be selected

* A reduced set of independent generators is characterized by the fact that the group generated by all except one of these generators has only the identity in common with the group generated by the remaining generators irrespective of the choice of this remaining generator.

for the first operator in the required reduced set. To find a second operator of the reduced set we select one of the remaining operators of the given set such that it must be raised to the highest possible power to be contained in the group generated by the determined operator of the required reduced set. The index of this highest power is clearly the order of a second independent generator of the group generated by the two selected operators, if the first of these operators is selected as one of its two independent generators. This second independent generator may then be used as the second operator of the required reduced set. This process is clearly similar to a well-known process to find a set of independent generators of an abelian group of order p^m, and can be continued until the λ operators of the required reduced set have been determined.

When the order of an abelian group is not a power of a prime number, the number of operators in a set of independent generators (reduced or non-reduced) may vary from the rank of the group to the sum of the ranks of its Sylow subgroups. By repeating the process explained in the preceding paragraph as many times as there are different prime divisors of the order of this abelian group, it results that we can find a reduced set of λ independent generators of any abelian group from any given set of λ independent generators. In speaking of abelian groups it is customary to use the term set of independent generators for *reduced* set of independent generators. The fact that non-abelian groups do not always possess a reduced set of independent generators may be illustrated by means of the Hamiltonian groups. It is evident that none of these groups has a reduced set of independent generators.

From the first paragraph of this section it is easy to derive a method to obtain the ϕ-subgroup of any abelian group of order g. In fact, it is only necessary to raise each operator of this abelian group to a power whose index involves all the prime factors of g but is not divisible by the square of one of these prime factors. To obtain a set of independent generators of the ϕ-subgroup of any abelian group of order g it is only necessary to raise each operator of a set of independent generators of the group to the given power and to reject those operators which reduce to the identity when they are raised to this power. *The order of the ϕ-subgroup of any abelian group of order g is therefore equal to g divided by the order of the subgroup generated by all the operators of prime orders contained in this abelian group.**

3. The ϕ-subgroups of the Sylow subgroups of the symmetric group

We shall begin with the case when G is a Sylow subgroup of order p^m contained in the symmetric group of degree p^n. Hence G is a transitive substi-

*Cf. G. Frattini, Atti della Reale Accademia dei Lincei, Rendiconti, ser. 4, vol. 2 (1886), p. 18.

tution group. In the special case when $\alpha = 2$ the value of m is clearly $p + 1$, and the order of the commutator subgroup of G is p^{p-1}, since a cyclic substitution of degree p is transformed under G into itself multiplied by a substitution consisting of two cycles of degree p. As the ϕ-subgroup of every group of order p^m includes the commutator subgroup of this group, it results directly that each of the Sylow subgroups of order p^m contained in the symmetric group of degree p^2 has exactly two independent generators.

It is well known that a Sylow subgroup of order p^m, contained in the symmetric group of degree p^α, may be constructed by forming the direct product of p Sylow subgroups of symmetric groups of degree $p^{\alpha-1}$ written on distinct sets of letters and extending this direct product by a substitution of order p which interchanges its p systems of intransitivity and transforms it into itself.* Hence it results directly that the index of the commutator subgroup of a Sylow subgroup of order p^m contained in the symmetric group of degree p^α is p times the index of the commutator subgroup of a Sylow subgroup of the symmetric group of degree $p^{\alpha-1}$. As the commutator subgroup of such a Sylow subgroup involves the pth power of every substitution of the group it results that the number of operators in a set of independent generators of a Sylow subgroup of order p^m contained in the symmetric group of degree p^α is always α.

It is now easy to determine the number of the operators in a set of independent generators of any Sylow subgroup of the symmetric group of degree n. In fact, if we write n in the form

$$n = a_1 p^{\alpha_1} + a_2 p^{\alpha_2} + \cdots + a_l$$

where $\alpha_1, \alpha_2, \cdots$ are positive integers in descending order of magnitude, and all of the coefficients $a_1, a_2, \cdots, a_l$ are positive integers which are less than p, then the Sylow subgroup of order p^m contained in the symmetric group of degree n is the direct product of a_1 transitive constituents of degree p^{α_1}, a_2 transitive constituents of degree p^{α_2}, etc. As the commutator subgroup of a direct product is the direct product of the commutator subgroups of the factors, it results from the preceding paragraphs that *the Sylow subgroup of order p^m contained in the symmetric group of degree*

$$n = a_1 p^{\alpha_1} + a_2 p^{\alpha_1-1} + \cdots + a_{\alpha_1} p + a_{\alpha_1+1}$$

has $m' = a_1 \alpha_1 + a_2 (\alpha_1 - 1) + \cdots + a_{\alpha_1}$ independent generators, where the coefficients $a_1, a_2, \cdots, a_{\alpha_1}$ are either 0 or positive integers less than p, and α_1 is a positive integer. The order of the ϕ-subgroup of a Sylow subgroup of order p^m contained in the symmetric group of degree n is therefore $p^{m-m'}$.

It results directly from what precedes that the ϕ-subgroup of an intransitive

* G. A. Miller, American Journal of Mathematics, vol. 23 (1901), p. 173.

Sylow subgroup of a symmetric group is the direct product of the ϕ-subgroups of its transitive constituents, and that the ϕ-subgroup of one of these transitive constituents of degree p^α, $\alpha > 1$, is an intransitive group with p transitive constituents which may be constructed as follows: Write a Sylow subgroup of the symmetric group of degree $p^{\alpha-1}$ on p distinct sets of letters, and construct an isomorphism between one of these p groups and each of the others in such a way that the $p - 1$ resulting groups of degree $2p^{\alpha-1}$ involve the direct products of the commutator subgroups of their transitive constituents, while the operators of their commutator quotient groups correspond to their inverses. The ϕ-subgroup of the given transitive group of degree p^α is the direct product of the $p - 1$ intransitive groups constructed in this manner.

4. Various sets of independent generators of any group

The number of ways of choosing a set of independent generators of a given group is generally very large. As regards an abelian group of order p^m, p being a prime number, it is well known that the number of possible choices of a reduced set of independent generators is equal to the order of the group of isomorphisms of this abelian group.* As regards non-abelian groups, in general, the matter becomes much more complex. In fact, the number of operators in the various possible reduced sets of independent generators is not necessarily the same since this number may evidently vary from 2 to $n - 1$ in the symmetric group of degree n, as was observed in the Introduction.

If a set of independent generators is so selected that the number of these generators is as large as possible it may be assumed that the order of each of these generators is a power of a prime number. That is, there is always at least one set of independent generators, involving the largest possible number of distinct operators, such that the order of each of these generators is a power of some prime number. In fact, if $s_1, s_2, \cdots, s_\lambda$ is a set of independent generators of G such that λ is a maximum, and if the order of s_α, $1 \leqq \alpha \leqq \lambda$, is not a power of a prime, it results that $s_\alpha = s'_\alpha \cdot s''_\alpha$ where it may be assumed that s'_α has an order which is a power of a prime and that the order of s''_α is prime to that of s'_α.

When at least one of the two groups generated by the two sets of λ operators, obtained by replacing successively s_α, in the given set, by s'_α and s''_α coincides with G the number of prime factors of the order of one of the independent generators in a set of λ independent generators of G has been reduced. If neither of these two groups would coincide with G we could replace s_α in the first set by the two operators s'_α and s''_α. Hence the given set would not have

*Cf. Bulletin of the American Mathematical Society, vol. 20 (1914), p. 360.

contained a maximum number of independent generators. The main result just proved may be embodied in a theorem as follows: *Among the possible sets of independent generators which involve a maximal number of operators there is at least one set in which the order of every operator is a power of some prime number.*

Let G be an intransitive substitution group and consider a set of independent generators of G such that this set involves the largest possible number of substitutions. If H is any transitive constituent of G, and if we select from the given set of generators of G any sub-set of operators which involve as their constituents from H a set of independent generators of H, then it is evident that all the other independent generators of G can be selected so as not to involve any letter from H. Moreover, this selection can always be so made as to preserve the number of operators in the sets of independent generators under consideration. Hence it results that *the number of operators in a set of independent generators of an intransitive substitution group cannot exceed the largest sum which can be obtained by adding the numbers of the operators in some one set of possible independent generators of each transitive constituent.*

It has been observed that the number of the operators in a set of independent generators of a group of order p^m is an invariant of the group. When this number is m the group is abelian and of type $(1, 1, \cdots)$. The other extreme case is when this number is unity and the group is cyclic. In each of these extreme cases there is only one possible group, but there exists clearly more than one possible group in every other case. In fact, there is at least one abelian and at least one non-abelian group in every other case. In the particular case when the number of these independent generators is $m - 1$ the possible abelian group has one invariant which is equal to p^2 while each of the other invariants is equal to p. The number of the possible non-abelian groups of order p^m which have $m - 1$ independent generators increases without limit as m increases.

ON A SET OF POSTULATES WHICH SUFFICE TO DEFINE A NUMBER-PLANE*

BY

ROBERT L. MOORE

In this paper I propose to show that any plane satisfying Veblen's Axioms I–VIII, XI of his *System of axioms for geometry*† is a number-plane, or in other words that any plane V satisfying these axioms contains a system of continuous curves such that, with reference to these curves regarded as straight lines, the plane V is an ordinary euclidean plane.

In consequence, any discussion of analysis situs based on these axioms (as, for example, Veblen's proof‡ of the theorem that a jordan curve divides its plane into just two parts) is no more general than one based on analytic hypotheses. This does not contradict the fact that the geometry based on axioms I–VIII, XI is much more general than euclidean geometry in the sense that the curves with respect to which the plane V is euclidean are not necessarily the straight lines referred to in these axioms.

My argument also furnishes the answer to a problem proposed by Veblen in another paper. He states this problem as follows.§

"An interesting situation is obtained by introducing a postulate of uniformity among the hypotheses of plane analysis situs (cf. p. 84 of this volume). If the postulate is applied to the straight line, the line is necessarily a continuum but it is not obvious that other curves are. If it is applied to the plane, the segments σ_{vP} in this case being triangular regions, all continuous curves are continua, but it is not obvious that there is a one-to-one reciprocal correspondence between the plane and a set of number-pairs."

Without adding any postulate of uniformity to Veblen's Axioms I–VIII,

* A part of this paper, under the title *Concerning so-called non-metrical analysis situs*, was presented to the Society, April 27, 1912.

† O. Veblen, *A system of axioms for geometry*, these Transactions, vol. 5 (1904), pp. 343–384.

‡ O. Veblen, *Theory of plane curves in non-metrical analysis-situs*, these Transactions, vol. 6 (1905), pp. 83–98. See also vol. 14, p. 65.

§ O. Veblen, *Definition in terms of order alone in the linear continuum and in well-ordered sets*, these Transactions, vol. 6 (1905), pp. 165–171.

XI, Lennes* has shown that all continuous curves in V are continua. He does not however show that there is a one-to-one reciprocal continuous correspondence between the plane V and the set of all real number pairs. In the present paper I establish the existence of such a correspondence. Under Theorem I, I show that every straight line in V is a linear continuum. This proof is quite different from Lennes' proof of the above-mentioned theorem concerning continuous curves.

THEOREM I. *Every straight line in V is a linear continuum.*

I will prove this theorem by showing that every such line l satisfies a set of five postulates for the linear continuum given by Veblen.† These postulates (A, B, C, D, E) are stated in terms of the relation $\prec$ (precedes). For the points of the line l this relation may be defined in an obvious way in terms of the order (betweenness) of Veblen's axioms I–VIII, XI. It is then easy to show that, on the line l, A, B, and D are satisfied. It remains to prove that C and E are satisfied. Postulates C and E are as follows.

C. PSEUDO-ARCHIMEDEAN POSTULATE 1. *If $\{P\}$ is an infinite set, then there exists a subset $[P_\nu]$, such that $P_\nu \prec P_{\nu+1}$ ($\nu = 1, 2, 3, \cdots$), and if P is any element of $\{P\}$ for which $P_1 \prec P$, then there exists a ν such that $P \prec P_\nu$.*

2. *If $\{P\}$ is an infinite set, then there exists a subset $[P_\nu]$, such that $P'_{\nu+1} \prec P'_\nu$ ($\nu = 1, 2, 3, \cdots$), and if P is any element of $\{P\}$ for which $P \prec P'$, then there exists a ν such that $P'_\nu \prec P$.*

E. POSTULATE OF UNIFORMITY. *For every element P of a set $\{P\}$ which satisfies conditions A, B, C, D, and for every ν ($\nu = 1, 2, 3, \cdots$) there exists a segment $\sigma_{\nu P}$ such that the set of segments $[\sigma_{\nu P}]$ has the following properties:*

1. *For a fixed P, $\sigma_{\nu P}$ contains $\sigma_{\nu+1 P}$.*

2. *For a fixed P, P lies on every $\sigma_{\nu P}$ and is the only such element.*

3. *For every segment τ, there exists a ν, ν_τ, such that for no P does $\sigma_{\nu_\tau P}$ contain τ.*

*Proof of C*1. On l there exist two points A and B such that $A \prec B$. There exists a point O not on l and a point T such that OT is parallel‡ to AB. The ray from T parallel to OA must intersect the ray AB in some point P_1. The ray from T parallel to OP_1 intersects AB in a point P_2. Continuing this process let $P_{\nu+1}$ denote the point in which AB is cut by the ray from T parallel

* N. J. Lennes, *Curves in non-metrical analysis situs with an application in the calculus of variations*, American Journal of Mathematics, vol. 33 (1911), pp. 285–326.

† O. Veblen, *Definition in terms of order alone, etc.*, loc. cit.

‡ In this paper the statement that OT is parallel to AB means that the *ray* OT is parallel to the *ray* AB. This signifies that OT itself fails to intersect AB but AB is intersected by every ray from O that goes into the angle TOA. For a proof that OT exists cf. Veblen, *A system of axioms for geometry*, loc. cit., p. 370. It is to be observed that we are not assuming the parallel postulate. There may be infinitely many lines through O which fail to cut the line AB.

to OP_ν. The set of points $[P_\nu]$ thus defined plays the part of $[P_\nu]$ in Veblen's $C1$. This may be seen as follows: Suppose P is any point such that $P_1 \prec P$. If there should exist no ν such that $P \prec P_\nu$ then it may be easily seen with the help of Postulate B* that there would exist a point P_0 such that

(1) for every ν, $P_\nu \prec P_0$,

(2) if $x \prec P_0$ then, for some ν, $x \prec P_\nu$.

There would exist a ray through O parallel to TP_0. This ray would cut AB in a point P' such that $P' \prec P_0$ and then there would exist ν_1 such that $P' \prec P_{\nu_1}$. It would clearly follow that $P_0 \prec P_{\nu_1+1}$, which is contrary to part (1) of the above definition of P_0. Thus the supposition that there exists no ν such that $P \prec P_\nu$ leads to a contradiction. Postulate C1 is therefore established. Postulate C2 may be proved in an entirely analogous manner.

Proof of Postulate E. There exist two distinct points A and B on the line l (Fig. 1). There exists a point O not on l. There exist two rays OC and OD

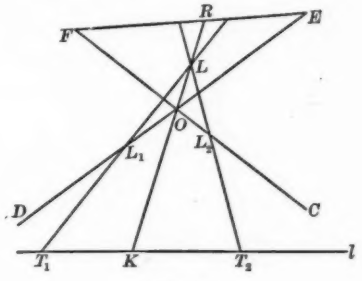

FIG. 1

parallel to AB and BA respectively. It may be easily proved with the aid of Postulate B that there exist, on the rays OC and OD respectively, two sets of points, $[P_\nu]$ and $[P'_\nu]$, such that, for every ν, $P_\nu P_{\nu+1} O$,† $P'_\nu P'_{\nu+1} O$, and such that Lim‡ $P_\nu = O$ and Lim $P'_\nu = O$. There exist points E and F such that DOE and COF. Let us suppose that OC and OD are not collinear.§ Then if P is any point on l the line PO meets the segment FE in a point S. It is clear that SP_ν and SP'_ν must cut l. Let S_ν and S'_ν respectively denote the points in which they cut it. Let $\sigma_{\nu P}$ denote the segment $S_\nu S'_\nu$. This segment plays the rôle of $\sigma_{\nu P}$ in Postulate E. That this is true as far as $E1$

* As stated above, Postulate B is a *theorem* here.

† ABC, *used as a sentence*, signifies that B is between A and C.

‡ Lim $P_\nu = O$ signifies that if k is any segment of the line OD containing O in its interior then there exists a positive number δ such that if $\nu > \delta$ then P_ν lies within k.

§ If OC and OD are collinear, replace the segment FE by the broken line FZE where Z is a point on the remote side of OC from l.

and $E2$ are concerned is evident. That $E3$ also is satisfied may be proved as follows. Let τ be any segment on l. Let T_1 and T_2 (where $T_1 \prec T_2$) denote its end points. There exists a point K such that $T_1 K T_2$. The line KO intersects the segment FE in a point R. There exists a point L between O and R. The segment $T_1 L$ intersects the ray OD in a point L_1 and $T_2 L$ intersects OC in a point L_2. Let ν_τ be the smallest ν for which P_ν lies between L_1 and O and P'_ν lies between L_2 and O. It is not difficult to see that this ν_τ plays the part of ν_τ in Postulate $E3$. Thus $E3$ is proved and Theorem I is established.

DEFINITION.* In the plane V the point P is said to be a limit point of the point set S if every triangle containing P in its interior contains at least one point of S different from P.

THEOREM II. *Between the plane V and an ordinary euclidean plane E there is a one-to-one reciprocal continuous† correspondence of points.*

Proof. Let l be a line lying in the plane V. By Theorem I the set of points that constitute this line may be brought into a one-to-one continuous correspondence with the set of all real numbers. Let a definite correspondence of this type be set up. The number which corresponds to a point on l will be called the abscissa of that point. Let O (Fig. 2) be the point whose abscissa

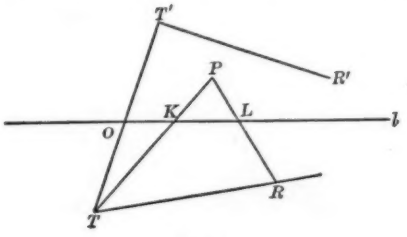

FIG. 2

is zero, and I the point whose abscissa is 1. There exists a point T not on l. There exists a point T' such that O is between T and T'. There exist rays TR and $T'R'$ parallel to OI. If P is any point on the opposite side of OI from T then PT and PR will intersect the line OI. Let K and L respectively denote the points in which they intersect it. The abscissa of P is defined as the abscissa of K and the ordinate of P is defined as $\overline{KL}$.‡ If P is on the opposite side of OI from T' then PT' and PR' will intersect OI in points K'

* Cf. O. Veblen, *Theory of plane curves in non-metrical analysis situs*, loc. cit., p. 85.

† The statement that such a correspondence is continuous signifies that if a point P in V is a limit point of a point set S lying in V, then P' is a limit point of S', and conversely, where P' and S' are the point and point set in E which correspond to P and S respectively.

‡ The symbol $\overline{KL}$ is used to denote the *number* obtained by subtracting the abscissa of K from that of L.

and L' respectively. In this case the abscissa of P is defined as the abscissa of K' but the ordinate of P is defined as $\overline{L'K'}$. In accordance with these definitions every point P in V has an abscissa x and an ordinate y.* It is not true however that to every real number pair (x, y) there corresponds a point in V whose coördinates are x and y. Indeed if P is a point on the remote side of OI from T then it is clear that the ray through R which is parallel to TP must cut OI in some point N and that $\overline{KN}$ is the least upper bound of the ordinates of all points whose common abscissa equals the abscissa of P. Now $\overline{KN}$ is a function of x, the abscissa of P. Let us write

$$\overline{KN} = f(x).$$

Similarly let $F(x)$ denote the greatest lower bound of the ordinates of all points with a given abscissa x which lie on the same side of OI as T. Now, using a rectangular system of coördinates in an ordinary euclidean plane, plot the two curves $y = f(x)$ and $y = F(x)$. The first of these curves will lie entirely above, and the second will lie entirely below, the x-axis. Let H denote the region lying between them. To each point P in V there corresponds the point of H whose rectangular coördinates are the V-coördinates of P. It is easy to see that the correspondence thus established between V and H is continuous. But H is clearly a simply connected region. Hence it may be brought into a one-to-one continuous correspondence with the whole plane E. It follows that there exists such a correspondence between V and E. Thus Theorem II is established.

In a definite correspondence of this type between V and E, each straight line in E corresponds to some continuous curve in V. It is clear that if the set S' of all curves in V that correspond to straight lines in E are themselves looked upon as straight lines, then

THEOREM III. *With respect to this new choice of straight lines V is an ordinary euclidean plane.*

The following is an equivalent statement in different form.

THEOREM IV. *In the plane V there exists a set S of simple continuous curves (each of which contains all its limit points) such that*

(1) *Each S-curve† divides the plane V into two parts.‡*

(2) *Through any two given points of V there is one and only one S-curve.*

(3) *If s is an S-curve and P is a point in V not lying on s then through P there is only one S-curve that fails to have a point in common with s.*

* Let us call these coördinates the *V-coördinates* of the point P.

† By an S-curve is meant a curve of the set S.

‡ The statement that the curve s divides the plane V into two parts signifies that the point-set $V - s$ consists of two mutually exclusive subsets V_1 and V_2 such that every two points in V_i $(i = 1, 2)$ can be joined by a jordan arc which contains no point of s, but no point of V_1 can be so joined to a point of V_2.

(4) *The set of curves S satisfies Desargues Theorem.**

It may be seen as follows that every set S satisfying the conditions stated in Theorem IV must also be a set S'† satisfying Theorem III. Suppose that s_1 and s_2 are two S-curves having a point O in common. Suppose that s_1 does not cross s_2 at O. Then it is clear‡ that O divides s_1 into two half-curves OA and OB and s_2 into two half-curves OC and OD such that OD and OA are separated from each other by OB and OC. The S-curve which joins A to D must then either cross the half-curve OB at a point E or cross the half-curve OC at a point F. The point E would, of necessity, be distinct from A and F would be distinct from D. Thus in one case the S-curve AD would have two distinct points A and E in common with s_1 and in the other case it would have two distinct points D and F in common with s_2. Hence both cases are contrary to condition (2) of Theorem IV. Therefore two S-curves with a point in common must *cross* each other at that point. It easily follows that Veblen's Triangle Transversal Axiom VIII§ holds true for the set of curves S. That his Axioms I–VII, XI, and XII hold true for S is clear. With the help of condition (4) of Theorem IV it follows‖ that the plane V may be regarded as lying in a 3-dimensional space R in which *all* of Veblen's Axioms I–XII are valid, the "straight lines" of R that lie in V being the S-curves. Hence the plane V is euclidean with respect to these curves as straight lines.

The *original* "straight lines" of V satisfy conditions (1) and (2) of Theorem IV, but they do not necessarily satisfy condition (3) or condition (4).¶

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* Cf. D. Hilbert, *Foundations of Geometry*, Townsend's translation, § 22.

† Cf. above.

‡ That no S-curve is closed can easily be proved with the aid of conditions (1) and (2).

§ O. Veblen, *A system of axioms for geometry*, loc. cit.

‖ Cf. Hilbert, loc. cit.

¶ Cf. Hilbert, loc. cit., and F. R. Moulton, *A simple non-desarguesian plane geometry*, these Transactions, vol. 3 (1902), p. 192.

THE EQUIVALENCE OF COMPLEX POINTS, PLANES, LINES WITH RESPECT TO REAL MOTIONS AND CERTAIN OTHER GROUPS OF REAL TRANSFORMATIONS*

BY

WILLIAM CASPAR GRAUSTEIN

1. Introduction. This paper has to do with geometry in three-dimensional *complex* space. The points, planes, lines of this space, the elements of the space in which we shall be interested, are called complex points, planes, lines, or briefly, complex elements. The special problem to be treated in the paper is the equivalence of these complex elements with respect to certain groups of *real* transformations, and in particular with respect to the group of real motions.

Two like-named complex elements are said to be *equivalent* with respect to a given group of transformations if there exists a transformation of the group carrying the one into the other.† Two such elements, which are equivalent with respect to the group of real motions in particular, we shall call, for the sake of brevity, *real-congruent*. Every two real like-named elements are of course real-congruent. But two imaginary‡ like-named elements are not, in general, real-congruent. To develop necessary and sufficient conditions for their real-congruence is the main object of this paper.

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* Presented to the Society, December 31, 1913, under a different title.

† The groups of real transformations, with respect to which we shall study the equivalence of complex elements, are (1) the group of real motions, (2) the group of real motions and reflections, (3) the group of real transformations of similitude. The equations in point coördinates of the group of $2 \cdot \infty^7$ transformations of similitude are

$$z_i' = r(a_i z_1 + b_i z_2 + c_i z_3 + d_i), \qquad r \neq 0, \qquad (i = 1, 2, 3),$$

where the a's, b's, c's, d's, and r are real and the determinant of the coefficients of the z's is orthogonal and has the value $+1$. These become the equations, (1) of the group of real motions and reflections, if we set $r = \pm 1$, (2) of the group of real motions alone, if we set $r = +1$.

We shall consider the equivalence of only those elements lying in the finite domain. Ruling out the elements at infinity causes no complications here, since all the groups of transformations considered are subgroups of the affine group.

‡ "Real" and "imaginary," as used here and throughout, are mutually exclusive; "complex" covers both.

2. Classification of points, planes, and lines.* A necessary condition that two complex points be real-congruent is that they be either both real or both imaginary. Accordingly, we divide complex points into two *classes*, the class of real points and the class of imaginary points.

That two complex planes (or lines) be real-congruent it is evidently necessary, *first*, that they contain the same number of real points; *second*, that the distribution of these real points between the finite and infinite domains be the same for both; *third*, that they should be both minimal or both euclidean (non-minimal).† Accordingly, a first step toward the solution of our problem will be a division into *classes* of complex planes and lines according to these three methods of distinction.

A complex plane contains either ∞^2 real points, in which case it is real, or ∞^1 real points, in which case it is imaginary. The ∞^1 real points of an imaginary plane are the real points of a real line, the line of intersection of the plane with its conjugate-imaginary plane. If this line is at infinity we shall call the plane *low-imaginary*; if the line is not at infinity we shall call the plane *high-imaginary*. (The direction of the normal of a low-imaginary plane is real, that of the normal of a high-imaginary plane is imaginary.) A low-imaginary plane is always euclidean. A high-imaginary plane, however, may be either euclidean or minimal.

Thus we have obtained *four* classes of planes, which we shall denote by distinctive capital letters, as follows:

Plane	Designation
Real	*R*-plane
Low-imaginary	*L*-plane
High-imaginary‡ minimal	*M*-plane
High-imaginary euclidean	*H*-plane

A complex straight line contains ∞^1 real points or lies in ∞^1 real planes when and only when it is real. An imaginary straight line either (*a*) contains one real point and lies in one real plane, or (*b*) contains no real point and lies in no real plane. In case (*a*) we shall call the line *incomplete-imaginary*; in case (*b*), *complete-imaginary*.

If the real point of an incomplete-imaginary line is at infinity, the line will be termed *low-imaginary*, otherwise *high-imaginary*. A low-imaginary line has a real direction and is therefore always euclidean.

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* The contents of this section appeared in greater detail in the author's dissertation, *Eine reelle Abbildung analytischer komplexer Raumkurven*, Bonn, 1913, pp. 4–6. For a discussion of the terminology see footnote (2), p. 4 of this dissertation.

† For the use of "euclidean" in the sense of "non-minimal" see Study, these Transactions, vol. 10 (1909), p. 19.

‡ Without causing ambiguity we may omit the adjective "high-imaginary" in the case of *M*-planes.

Finally we combine the division (of the lines with imaginary directions) into high- and complete-imaginary with the division into euclidean and minimal.

Thus we have in all six *classes* of lines, to be designated by suggestive capital letters, as follows:

LINE	DESIGNATION
Real	*R*-line
Low-imaginary	*L*-line
Incomplete-imaginary minimal	*M*-line
High-imaginary euclidean	*H*-line
Complete-imaginary minimal	*CM*-line
Complete-imaginary euclidean	*CE*-line

3. Fundamental theorem. We introduce the following notation. The element which is conjugate-imaginary to a given element T shall be denoted by $\overline{T}$, the (shortest) distance from T to $\overline{T}$ by $d(T, \overline{T})$, and the angle between T and $\overline{T}$, measured from T to $\overline{T}$, if it exists, by $\measuredangle(T, \overline{T})$. As is well known, $d^2(T, \overline{T})$ and, when existent, $\tan^2\measuredangle(T, \overline{T})$ are absolute invariants of T and $\overline{T}$ under the group of real motions.*

FUNDAMENTAL THEOREM: *The necessary and sufficient conditions that two like-named imaginary elements, R and S, be real-congruent are* (1) *that they belong to the same class,* (2) *that* $d^2(R, \overline{R}) = d^2(S, \overline{S})$, *and* (3), *if* $\measuredangle(R, \overline{R})$ *and* $\measuredangle(S, \overline{S})$ *exist, that* $\tan^2\measuredangle(R, \overline{R}) = \tan^2\measuredangle(S, \overline{S})$.†

The conditions of the theorem are evidently necessary. In proving their sufficiency—the crux of the whole problem—we propose to establish first the theorem analogous to the Fundamental Theorem for each class of complex elements of (*a*) the real line, and (*b*) the real plane. These theorems, besides being of interest in themselves, will serve as the basis for the proof of the sufficiency of the Fundamental Theorem.

4. Cases of one and two dimensions. For the one-dimensional geometry of the real line "real-congruence" signifies equivalence with respect to the group of ∞^1 real translations of the line into itself.

THEOREM 1. *Two real points are always real-congruent under one and only one translation.*

THEOREM 2. *If the two imaginary points, A and B, are real-congruent, then* $d(A, \bar{A}) = d(B, \bar{B})$.‡ *Conversely, if* $d(A, \bar{A}) = d(B, \bar{B})$, *$A$ is carried into B by one and only one real translation.*

* An analytical method for setting up these invariants is given in § 7.

† For the complete-imaginary lines a slight change must be made in condition (2); see Theorems 19, 20. It is also to be noted that either of the conditions (2) or (3) may, in a given case, be no longer restrictive: condition (2) if R and S are members of a class for which $d^2(T, \overline{T}) = 0$ (i. e., if R and S are M- or H-planes, or M- or H-lines); condition (3) if R and S belong to a class for which $\tan^2\measuredangle(T, \overline{T}) = 0$ (i. e., if R and S are L-planes or L-lines). Finally the condition (3) drops naturally if R and S are points or minimal elements.

‡ The algebraic and not the absolute value of d is meant here, since directed distance is an absolute invariant of translations of a line into itself. Consequently, two conjugate-imaginary points are not equivalent with respect to real translations of their line into itself.

For if $d(A, \bar{A}) = d(B, \bar{B})$, the translation carrying the (real) midpoint of A and $\bar{A}$ into the (real) midpoint of B and $\bar{B}$ carries A into B. The analytical proof is also simple.

The real plane contains, of the elements in which we are interested, R-, L-, M-, and H-lines, besides points. "Real-congruence" signifies here equivalence with respect to the group of ∞^3 real motions of the plane into itself.

THEOREM 3. *Two real points, A and B, are always real-congruent under ∞^1 motions* (the translation carrying A into B, combined with the ∞^1 rotations about B).

THEOREM 4. *Two real lines, a and b, are always real-congruent under $2 \cdot \infty^1$ motions* (two rotations* of a into b, combined with ∞^1 real translations along b).

THEOREM 5. *If two imaginary points, A and B, are real-congruent,*

$$d^2(A, \bar{A}) = d^2(B, \bar{B}).$$

Conversely, if $d^2(A, \bar{A}) = d^2(B, \bar{B})$, A is carried into B by one and only one real motion.

For if $d^2(A, \bar{A}) = d^2(B, \bar{B})$, then, of the $2 \cdot \infty^1$ real motions carrying the real line $A\bar{A}$ into the real line $B\bar{B}$, ∞^1 will have the effect that, in this new position, $d(A, \bar{A}) = d(B, \bar{B})$, and of these motions one and only one will carry A into B (cf. Theorem 2).

THEOREM 6. *Two L-lines, a and b, are real-congruent only when*

$$d^2(a, \bar{a}) = d^2(b, \bar{b}),$$

and then under ∞^1 motions.

For if $d^2(a, \bar{a}) = d^2(b, \bar{b})$, then, of the $2 \cdot \infty^1$ motions carrying the (real) line midway between a and $\bar{a}$ into the (real) line midway between b and $\bar{b}$ (cf. Theorem 4), ∞^1 will carry a into b.

THEOREM 7. *Two M-lines, a and b, are real-congruent only when they go through the same circular point at infinity and then under ∞^1 motions* (the translation of the real point of a into the real point of b, combined with the ∞^1 real rotations about the latter point).

THEOREM 8. *If two H-lines, a and b, are real-congruent,*

$$\tan \tfrac{1}{2}(a, \bar{a}) = \tan \tfrac{1}{2}(b, \bar{b}).\dagger$$

Conversely, if $\tan \frac{1}{2}(a, \bar{a}) = \tan \frac{1}{2}(b, \bar{b})$, then there are two and only two real motions carrying a into b.

* If a and b are parallel, the two rotations are to be replaced by a translation of a into b and a combination of this translation and a rotation through 180° of b about any of its points.

† Tangent of the angle and not the square of the tangent, since oriented angle is an absolute invariant of motions in a plane. Hence two conjugate-imaginary H-lines are not equivalent with respect to the real motions in their plane. This is also true for two conjugate-imaginary M-lines.

Translate the point of intersection of a and $\bar{a}$ into the point of intersection of b and $\bar{b}$. Then either of the two rotations carrying the real lines bisecting the angles between a and $\bar{a}$ into the corresponding bisectors of the angles between b and $\bar{b}$ will carry a into b, provided that $\tan \measuredangle(a, \bar{a}) = \tan \measuredangle(b, \bar{b})$.*

5. Case of three dimensions. We now proceed to the proof of the Fundamental Theorem for each class of complex elements of three-dimensional space.

THEOREM 9. *Two real points, A and B, are always real-congruent under ∞^3 motions* (the translation carrying A into B, combined with the ∞^3 real rotations about B).

THEOREM 10. *Two real planes, α and β, are always real-congruent under $2 \cdot \infty^3$ motions* (two rotations† of α into β about their line of intersection, combined with ∞^3 motions leaving β invariant).

THEOREM 11. *Two real lines, a and b, are always real-congruent under $2 \cdot \infty^2$ motions* (a motion carrying a into b, combined with the $2 \cdot \infty^1$ rotations and ∞^1 translations leaving b invariant).

THEOREM 12. *If two imaginary points, A and B, are real-congruent,*

$$d^2(A, \bar{A}) = d^2(B, \bar{B}).$$

Conversely, if $d^2(A, \bar{A}) = d^2(B, \bar{B})$, A is carried into Bb y ∞^1 real motions.

The proof is similar to that of Theorem 5.

THEOREM 13. *Two L-planes, α and β, are real-congruent only when*

$$d^2(\alpha, \bar{\alpha}) = d^2(\beta, \bar{\beta}),$$

and then under ∞^3 motions.

The proof is similar to that of Theorem 6.

THEOREM 14. *Any two M-planes, α and β, are real-congruent under ∞^2 motions.*

Of the $2 \cdot \infty^2$ real motions carrying the line of intersection of α and $\bar{\alpha}$ into the line of intersection of β and $\bar{\beta}$ (cf. Theorem 11), ∞^2 will carry α into β.

THEOREM 15. *If two H-planes, α and β, are real-congruent,*

$$\tan^2 \measuredangle(\alpha, \bar{\alpha}) = \tan^2 \measuredangle(\beta, \bar{\beta}).$$

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* An analytical proof follows. Translate the real point of a into the real point of b, and then take this point as origin of coördinates. The equations of a and b are respectively

$$\mu x - iy = 0 \quad \text{and} \quad \nu x - iy = 0.$$

The necessary and sufficient condition that a and b be real-congruent is that the tangent of the angle between a and b, namely $i(\mu - \nu)/(1 - \mu\nu)$, be real; this results in

$$(\mu + \bar{\mu})/(1 + \mu\bar{\mu}) = (\nu + \bar{\nu})/(1 + \nu\bar{\nu}),$$

where $\bar{\mu}, \bar{\nu}$ are conjugate-complex to μ, ν respectively. But $\tan \measuredangle(a, a) = i(\mu + \bar{\mu})/(1 + \mu\bar{\mu})$, and $\tan \measuredangle(b, \bar{b}) = i(\nu + \nu)/(1 + \nu\nu)$.

† If the planes are parallel the two rotations are to be replaced by a translation of the first into the second and a combination of this translation and a rotation through 180° about any real line in the second.

Conversely, if $\tan^2 \measuredangle(\alpha, \bar{\alpha}) = \tan^2 \measuredangle(\beta, \bar{\beta})$, *then* α *is carried over into* β *by* $2 \cdot \infty^1$ *real motions.*

For if $\tan^2 \measuredangle(\alpha, \bar{\alpha}) = \tan^2 \measuredangle(\beta, \bar{\beta})$, we may carry the real line of α into the real line of β so that in this position $\tan \measuredangle(\alpha, \bar{\alpha}) = \tan \measuredangle(\beta, \bar{\beta})$. Proceed then as in Theorem 8. Finally, any (real) translation in the direction of the real line of β will leave α and β coincident.

THEOREM 16. *Two L-lines,* a *and* b, *are real-congruent only when*

$$d^2(a, \bar{a}) = d^2(b, \bar{b}),$$

and then under $2 \cdot \infty^1$ *motions.*

Rotate the plane of a and $\bar{a}$ into the plane of b and $\bar{b}$ about their line of intersection in either of the two possible ways.* Then apply Theorem 6.

THEOREM 17. *Two M-lines,* a *and* b, *are always real-congruent under* ∞^1 *motions.*

Carry the plane of a and $\bar{a}$ into the plane of b and $\bar{b}$ so that a and b will go through the same circular point at infinity. Then apply Theorem 7.

THEOREM 18. *If two H-lines,* a *and* b, *are real-congruent,*

$$\tan^2 \measuredangle(a, \bar{a}) = \tan^2 \measuredangle(b, \bar{b}).$$

Conversely, if $\tan^2 \measuredangle(a, \bar{a}) = \tan^2 \measuredangle(b, \bar{b})$, a *is carried into* b *by two and only two real motions.*

For if $\tan^2 \measuredangle(a, \bar{a}) = \tan^2 \measuredangle(b, \bar{b})$, we may carry the plane of a and $\bar{a}$ into the plane of b and $\bar{b}$ so that in this position $\tan \measuredangle(a, \bar{a}) = \tan \measuredangle(b, \bar{b})$. Then apply Theorem 8.

In order to treat the case of CE-lines, we associate with any given CE-line a a certain N-plane N_a and a certain H-line H_a. N_a is, namely, the only plane through a having a normal with a real direction; H_a is the projection of a upon the real plane midway between N_a and $\bar{N}_a$. H_a is parallel to a, and hence $\tan \measuredangle(a, \bar{a}) = \tan \measuredangle(H_a, \bar{H}_a)$. The common perpendicular to a and $\bar{a}$ is the real normal to the real plane of H_a in the real point of H_a; thus it is normal to both N_a and $\bar{N}_a$ and hence $d(a, \bar{a}) = d(N_a, \bar{N}_a)$. This distance, as given by the formula† below, is double valued. In order to obtain a singled-valued, i. e., a directed distance, we must assign a determination to a certain radical. This radical occurs also in the expressions for the direction cosines of the common perpendicular to a and $\bar{a}$. We orient this perpendicular‡ and thereby direct the distance $d(a, \bar{a})$. It is this

* See preceding footnote.

† Cf. J in the list in § 7.

‡ This we do as follows. If the direction cosines of a are proportional to $p_1 : p_2 : p_3$, those of $\bar{a}$ are proportional to $p_1 : \bar{p}_2 : \bar{p}_3$ and those of the common perpendicular to

$$i(p_2 \bar{p}_3 - p_3 \bar{p}_2) : i(p_3 p_1 - p_1 \bar{p}_3) : i(p_1 \bar{p}_2 - p_2 \bar{p}_1).$$

We orient this perpendicular by choosing as its direction cosines these quantities each divided by the positive square root of the sum of their squares.

directed distance that we shall understand henceforth by d, when d is applied to CE-lines.

THEOREM 19. *Two CE-lines, a and b, are real-congruent only when*

$$d(a, \bar{a}) = d(b, \bar{b}) \qquad \textit{and} \qquad \tan^2 \measuredangle (a, \bar{a}) = \tan^2 \measuredangle (b, \bar{b}),$$

and then under two and only two motions.

From the preceding considerations it is evident that a real motion carrying a into b must carry H_a into H_b. But there are two and only two real motions carrying H_a into H_b, if $\tan^2 \measuredangle (a, \bar{a}) = \tan^2 \measuredangle (b, \bar{b})$; for then

$$\tan^2 \measuredangle (H_a, \bar{H}_a) = \tan^2 \measuredangle (H_b, \bar{H}_b),$$

and we may apply Theorem 18. Moreover these motions carry the directed common perpendicular to a and $\bar{a}$ into the directed common perpendicular to b and $\bar{b}$. Now if $d(a, \bar{a}) = d(b, \bar{b})$, then $d(N_a, \bar{N_a}) = d(N_b, \bar{N_b})$, and hence either of the motions carries N_a into N_b and therefore a into b.

The case of the CM-lines may be treated similarly; or in this treatment of CE-lines, replace CE by CM, H by M, use Theorem 17 instead of Theorem 18 in the proof and omit the statements involving angle. We give merely the theorem.

THEOREM 20. *Two CM-lines, a and b, are real-congruent only when*

$$d(a, \bar{a}) = d(b, \bar{b}),$$

and then under ∞^1 motions.

6. Extension to transformations of symmetry and similitude. We note that, whenever directed distance or angle is an absolute invariant with respect to the group of real motions, it is a relative invariant with -1 as the factor of reproduction with respect to the corresponding *set of real reflections*, and that, whereas any motion in a real plane leaves each of the minimal directions in this plane invariant, the corresponding reflection interchanges the two minimal directions. If we make changes in accordance with these facts in Theorems 1–20, we obtain necessary and sufficient conditions for equivalence with respect to the set of real reflections.

It is now evident that, if we combine the *real motions and reflections* into a single group, we may obtain valid theorems for equivalence with respect to this group by replacing, in Theorems 1–20, every equality linear in d or in the tangent of an angle by the corresponding quadratic equality and by omitting the condition of Theorem 7. Then the number of transformations of the mixed group carrying the first element into the second is, in the case of the altered Theorems 2, 7, 8, 19, 20, the same as the number of real motions effecting the result, but in the case of the other theorems, double* this number.

* That is, if $k \cdot \infty^a$ is the number of motions, the number of motions and reflections is $2k \cdot \infty^a$.

Finally we recall that, with respect to the group of *real transformations of similitude*, d^2 is but a relative invariant. (The square of the tangent of an angle remains an absolute invariant.) Accordingly, to obtain necessary and sufficient conditions for equivalence under this group, we omit in Theorems 1–20 all conditions involving distance (besides, of course, the condition of Theorem 7) and replace $\tan \measuredangle (a, \bar{a}) = \tan \measuredangle (b, \bar{b})$ in Theorem 8 by

$$\tan^2 \measuredangle (a, \bar{a}) = \tan^2 \measuredangle (b, \bar{b}).$$

The number of real transformations of similitude carrying the first element into the second is in every case ∞^1 times* the number of real motions and reflections effecting the result.

7. A complete system of rational invariants. During the remainder of the paper we shall restrict our considerations to three-dimensional space. Analogous work for the real line and the real plane is so simple as hardly to merit explicit treatment here and at the same time is, for the most part, deducible from the discussion of the three dimensional case.

The conditions given in Theorems 9–20 for the real-congruence of complex elements are based on absolute invariants of these elements under the group of real motions. We now give the analytical expression for each of these invariants (also the interval of values it takes on for the class for which it is given).†

Imaginary points $Z \equiv (\overline{z - \bar{z}} \,|\, \overline{z - \bar{z}}) = d^2(z, \bar{z}), \qquad -\infty < Z < 0,$

L-planes $\quad L_p \equiv \dfrac{w_0^2}{(w|w)} - \dfrac{2w_0\,\bar{w}_0}{(w|\bar{w})} + \dfrac{\bar{w}_0^2}{(\bar{w}|\bar{w})} = d^2(w, \bar{w}), \qquad -\infty < L_p < 0,$

H-planes $\quad I_p \equiv \dfrac{(w\bar{w}\,|\,w\bar{w})}{(w|\bar{w})^2} = \tan^2 \measuredangle (w, \bar{w}), \qquad -1 < I_p < 0,$

L-lines $\quad L_l \equiv \dfrac{(q|q)}{(p|p)} - 2\dfrac{(q|\bar{q})}{(p|\bar{p})} + \dfrac{(\bar{q}|q)}{(\bar{p}|\bar{p})} = d^2(p : q, \bar{p} : q), \qquad -\infty < L_l < 0,$

H-lines $\quad I_l \equiv \dfrac{(p\bar{p}\,|\,p\bar{p})}{(p|\bar{p})^2} = \tan^2 \measuredangle (p : q, \bar{p} : q), \qquad -1 < I_l < 0,$

* That is, if $k \cdot \infty^0$ is the number of motions and reflections, the number of transformations of similitude is $k \cdot \infty^{0+1}$.

† Here and later we use z_1, z_2, z_3 as the coördinates of a point z, and w_0, w_1, w_2, w_3 as the homogeneous coördinates of a plane w; also $p_1, p_2, p_3; q_1, q_2, q_3$ as the plueckerian coördinates of a line $p : q$. Further, if $r : (r_1, r_2, r_3)$, $s : (s_1, s_2, s_3)$, etc., are any triples of numbers, we denote the triple $(r_2 s_3 - r_3 s_2, r_3 s_1 - r_1 s_3, r_1 s_2 - r_2 s_1)$ by $\hat{rs}$, and introduce the symbols $(r|s)$, $(rs|rs)$, (rsl), where

$$(r|s) = r_1 s_1 + r_2 s_2 + r_3 s_3, \qquad (rs|rs) = (\hat{rs}\,|\,\hat{rs}), \qquad (rsl) \equiv \begin{vmatrix} r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \\ l_1 & l_2 & l_3 \end{vmatrix}.$$

CM-lines $$J \equiv i\frac{(p|\bar{q}) + (\bar{p}|q)}{\sqrt{-(p\bar{p}|p\bar{p})}} = d(p:q,\ \bar{p}:\bar{q}), \qquad 0 < J^2 < \infty,$$

CE-lines I and J $\qquad -1 < I_l < 0,\ 0 < J^2 < \infty.$

To derive these expressions one has but to evaluate d, d^2 and $\tan^2\measuredangle$ in the several cases. But since the expressions are absolute invariants under the group of real motions, they should be obtainable from the equations of this group. We proceed to derive I_l, J, and L_l in this way; the derivation of the others is similar.

The equations of the group of real motions in line coördinates are

$$\rho(p'|\omega) \underset{\omega}{\equiv} (a|\omega)p_1 + (b|\omega)p_2 + (c|\omega)p_3, \quad (\rho, \text{complex}, \neq 0),$$

$$\rho(q'|\omega) \underset{\omega}{\equiv} (da\omega)p_1 + (db\omega)p_2 + (dc\omega)p_3 + (bc\omega)q_1 + (ca\omega)q_2 + (ab\omega)q_3. \dagger \tag{1}$$

From the first identity we have

$$\rho^2(p'|p') \equiv (p|p), \tag{2}$$

$$\rho\bar{\rho}(p'|\bar{p}') \equiv (p|\bar{p}). \tag{3}$$

We may combine (2) and (3) into

$$(\rho\bar{\rho})^2(p'\,\bar{p}'|p'\,\bar{p}') \equiv (p\bar{p}|p\bar{p}). \tag{4}$$

From (3) and (4) we then have

$$I_l \equiv \frac{(p'\,\bar{p}'|p'\,p')}{(p'|\bar{p}')^2} \equiv \frac{(p\bar{p}|p\bar{p})}{(p|\bar{p})^2}.$$

Using both identities of (1) we obtain

$$\rho\bar{\rho}[(p'|\bar{q}') + (\bar{p}'|q')] \equiv (p|\bar{q}) + (\bar{p}|q). \tag{5}$$

From (4) and (5) we may now show that J is an absolute invariant; but in doing this we must rule out the lines with real directions, since for them $(p\bar{p}|p\bar{p}) = 0$. Thus J is defined only for lines with imaginary directions and has the further disadvantage of being irrational. Now dividing (5) by (3) yields the absolute invariant

$$K \equiv \frac{(p|q) + (p|\bar{q})}{(p|\bar{p})},$$

which has neither of these drawbacks. We use it from now on in preference to J.

* We take the positive value of this square root.

† Each of these identities in ω, i. e., in ω_1, ω_2, ω_3, represents three equations, namely those obtained by equating the coefficients of the two members of the identity for $i = 1, 2, 3$.

To deduce L_l we must first restrict $p : q$ to be a line with a real direction. This done, the left-hand members of the two identities,

$$p_1(da\omega) + p_2(db\omega) + p_3(dc\omega) \equiv_\omega \rho(q'|\omega) - (bc\omega)q_1 - (ca\omega)q_2 - (ab\omega)q_3,$$

$$\bar{p}_1(da\omega) + \bar{p}_2(db\omega) + \bar{p}_3(dc\omega) \equiv_\omega \bar{\rho}(\bar{q}'|\omega) - (bc\omega)\bar{q}_1 - (ca\omega)\bar{q}_2 - (ab\omega)\bar{q}_3,$$

are proportional. Eliminating the left hand members and substituting for ρ and $\bar{\rho}$ their values from (2), we have

$$\frac{(q'|\omega)}{\sqrt{p'|p'}} - \frac{(\bar{q}'|\omega)}{\sqrt{\bar{p}'|\bar{p}'}} \equiv_\omega (bc\omega)\left[\frac{q_1}{\sqrt{p|p}} - \frac{q_1}{\sqrt{\bar{p}|\bar{p}}}\right] + (ca\omega)\left[\frac{q_2}{\sqrt{p|p}} - \frac{\bar{q}_2}{\sqrt{\bar{p}|\bar{p}}}\right] + (ab\omega)\left[\frac{q_3}{\sqrt{p|p}} - \frac{\bar{q}_3}{\sqrt{\bar{p}|\bar{p}}}\right],$$

holding for proper determinations of the radicals. From this identity we obtain L_l as an absolute invariant.

In the deduction of I_l and K, $p : q$ was unrestricted. Hence I_l and K are defined, i. e., have meaning as invariants, for every class of lines. Similar statements hold for Z and I_p. On the other hand L_l is defined (as an invariant) only when $p : q$ is a line of real direction; similarly for L_p.

Now in the list given each invariant was given only for a particular class (or classes) and the (open) interval of the values it takes on just for this class (or classes) was added. For the other classes for which it is defined, the invariant has the constant value (or values) represented by the finite end point (or points) of this interval. Thus I_p is listed only for the class of H-planes, for which $-1 < I_p < 0$; but for all M-planes $I_p = -1$ and for all R- and L-planes $I_p = 0$.*

On the basis of Theorems 9–20 we conclude that the invariant (or invariants) given for any class of complex elements forms (or form) *a complete system of absolute rational invariants* of the general element† of this class under the group of real motions. [J (irrational) is supposed to have been replaced

* For the values of any invariant for each of the classes of elements for which it is defined, see the tables in § 8.

† Or better, from the point of view of *analytical* invariants, of a pair of conjugate-imaginary elements of the class. For there exists no *analytical* invariant of a single complex element, i. e., no expression, rational in the coördinates of a single complex element, which is an invariant. Suppose, for instance, that $f(z_1, z_2, z_3)$ were an invariant of the complex point z. Since z and $\bar{z}$ are real-congruent, $f(z_1, z_2, z_3) \equiv f(\bar{z}_1, \bar{z}_2, \bar{z}_3)$, whence follows that f must be constant. Similar reasoning shows that an invariant of a pair of conjugate-imaginary elements is a *symmetrical* function of the two sets of coördinates; consequently we may choose such invariants so that they will be everywhere *real-valued*.

These considerations bring into prominence a fact which was already evident, namely that, in solving the problem of the real-congruence of two complex elements, we have really solved a slightly more general problem, namely that of the real-congruence of two ordered pairs of conjugate-complex elements.

by K.] For if there existed an (absolute) rational invariant of a complex element not expressible in terms of the listed invariant (or invariants) of the class to which this element belongs (and, of course, not becoming trivial for the class, i. e., not taking on a constant value for all its members), then two elements of the class would not be real-congruent unless they had the same value for this new invariant; this contradicts the theorem on real-congruence for the class in question.

8. Summary and tabulation of results. We shall now subdivide each class of elements into *subclasses*, where a *subclass* is the totality of all elements of a class for which the (absolute) invariants defined for the class have the same value. Since the invariants defined for any class not listed in the table in § 7 have constant values for all elements of the class, such a class contains but one subclass, namely itself. Of the classes given in the table, that of the CE-lines contains a double infinity of subclasses and each of the remaining ones a single infinity of subclasses.

We may now restate the Fundamental Theorem (§ 3) in better form.

FUNDAMENTAL THEOREM. *A necessary and sufficient condition that two like-named complex elements be real-congruent is that they belong to the same subclass.*

The following tables give a summary of our results in the form of a division, first of the complex elements into classes and then of the classes into subclasses; the last column gives the number of motions under which two elements of one of the subclasses of the class in question are real-congruent.

POINTS

Class	*No. of points in class*	*Value for Z*	*No. of subclasses*	*No. in one subclass*	*No. of motions*
Real	∞^3	0	One	∞^3	∞^3
Imaginary	∞^6	$\neq 0$	∞^1	∞^5	∞^1

PLANES

Class	*No. of planes in class*	*Value for L_p*	*Value for I_p*	*No. of subclasses*	*No. in one subclass*	*No. of motions*
R	∞^3	0	0	One	∞^3	$2\cdot\infty^3$
L	∞^4	$-\infty < L_p < 0$	0	∞^1	∞^3	∞^3
M	∞^4		-1	One	∞^4	∞^2
H	∞^6		$-1 < I_p < 0$	∞^1	∞^5	$2\cdot\infty^1$

LINES

Class	*No. of lines in class*	*Value for L_l*	*Value for I_l*	*Value for K*	*No. of subclasses*	*No. in one subclass*	*No. of motions*
R	∞^4	0	0	0	One	∞^4	$2\cdot\infty^2$
L	∞^6	$\neq 0$	0	0	∞^1	∞^5	$2\cdot\infty^1$
M	∞^5		-1	0	One	∞^5	∞^1
H	∞^7		$-1 < I_l < 0$	0	∞^1	∞^6	2
CM	∞^6		-1	$\neq 0$	$2\cdot\infty^1$	∞^5	∞^1
CE	∞^8		$-1 < I_l < 0$	$\neq 0$	$2\cdot\infty^2$	∞^6	2

Inspection of the last two columns in each table shows us that the number of dimensions of the manifold of complex elements forming a subclass plus the number of dimensions of the manifold of real motions, under which two elements of the subclass are real-congruent, is always equal to six, the number of dimensions of the group of real motions.

Similar tables (from which we may draw similar conclusions) may be constructed to exhibit the results of § 6 for the equivalence of complex elements with respect to (1) the group of real motions and reflections, (2) the group of real transformations of similitude. In this connection we note that, of the absolute invariants under the group of real motions, only I_p and I_l are absolute invariants under the group of real transformations of similitude. For the others we have

$$Z' = r^2 Z, \qquad L'_p = r^2 L_p, \qquad L'_l = r^{-2} L_l, \qquad K' = rK.*$$

For the group of real motions and reflections, $r = \pm 1$; hence all but K are absolute invariants under this group and K is replaced by K^2.

HARVARD UNIVERSITY,
CAMBRIDGE, MASSACHUSETTS,
March, 1914.

* For the meaning of r, see the second footnote of the paper.

INVARIANTS OF THE RATIONAL PLANE QUINTIC CURVE AND OF ANY RATIONAL CURVE OF ODD ORDER*

BY

JOSEPH EUGENE ROWE

1. **Introduction.** The subject of *Rational Curves* is a growing one, and it is desirable to have in our mathematical literature a maximum of information in regard to them in a minimum of printed pages. Invariants of rational curves are rather long expressions and instead of encumbering a journal with such expressions it is much better to use if possible some expression which already occurs in a convenient place. In my dissertation, a piece of work done under Dr. Morley's direction, I have shown among other things how certain combinants of two binary quartics, explicitly written out in Salmon,† may be interpreted as invariants of the rational plane quartic which I call the R^4. In this paper I am going to develop a method by which these same combinants may be interpreted as invariants of any rational plane curve of odd order, and the process will be carried out in detail for the R^5 as an illustrative example and because of the special interest which invariants of the plane R^5 possess.

2. **Transvectants of Line Sections.** Let the parametric equations of a rational plane curve of order n, which I shall refer to as the R^n, be

$$x_i = (\alpha_i t)^n \qquad (i = 0, 1, 2), \tag{1}$$

in symbolic expressions which are the equivalent of

$$x_i = a_i t^n + C_1^n b_i t^{n-1} + C_2^n c_i t^{n-2} \cdots \qquad (i = 0, 1, 2). \tag{2}$$

If (1) is cut by the two lines

$$(\xi x) = \xi_0 x_0 + \xi_1 x_1 + \xi_2 x_2 = 0, \tag{3}$$

and

$$(\eta x) = \eta_0 x_0 + \eta_1 x_1 + \eta_2 x_2 = 0, \tag{4}$$

we obtain two binary n-ics which may be written symbolically

$$(\beta t)^n = 0, \tag{5}$$

$$(\gamma t)^n = 0. \tag{6}$$

* Presented to the Society, April 25, 1914.

† *Higher Algebra*, Fourth Edition, pp. 219–222.

The expression

$$(\beta\gamma)^r(\beta t)^{n-r}(\gamma t)^{n-r} = 0 \qquad (n \leqq r) \tag{7}$$

is a function of (5) and (6) and is called the rth transvectant of (5) and (6). If x_0, x_1, x_2 are substituted for the coördinates of the point in which (3) and (4) intersect, these transvectants may be expressed as binary forms in t of degree $2(n-r)$ whose coefficients are linear functions of the three-rowed determinants of the type

$$\begin{vmatrix} a_0 & b_0 & x_0 \\ a_1 & b_1 & x_1 \\ a_2 & b_2 & x_2 \end{vmatrix} \equiv |abx|, \tag{8}$$

so long as r is an odd number. For instance, if $r = 1$, (7) becomes the tangential equation of the R^n, that is, it becomes a binary form equated to zero in which, if a particular value of t, say t_1, is substituted, the result is the equation of the tangent to the R^n at the point whose parameter is t_1; but if the coördinates of a particular point are inserted for x_0, x_1, x_2, the resulting binary form equated to zero has $2(n-1)$ roots which are the parameters of the $2(n-1)$ tangents to the R^n through this particular point.

In particular, if $r = n$ in (7), that expression becomes

$$(\beta\gamma)^n = 0. \tag{9}$$

For any odd value of n this becomes, by means of the scheme described above, the equation of *a* covariant line of the corresponding R^n, and it will be referred to as *the* covariant line of the R^n since it has the distinction of being its simplest covariant line. An idea of the appearance of the equations of these lines may be formed by carrying out in detail the process just indicated for a few of the R^n. I give several below, indicating the particular R^n and the corresponding covariant line L_n.

$$R^3 \text{ has } L_3 \equiv |adx| - 3|bcx| = 0. \tag{10}$$

$$R^5 \text{ has } L_5 \equiv |afx| - 5|bcx| + 10|cdx| = 0. \tag{11}$$

$$R^7 \text{ has } L_7 \equiv |abx| - 7|bgx| + 21|cfx| - 35|dex| = 0. \tag{12}$$

The equations of others may be written out from analogy with these.

3. **Osculants of R^n.** The equations

$$x_i = (a_i t')^s(\alpha_i t)^{n-s} \qquad (i = 0, 1, 2), \tag{13}$$

where t' is constant, define a rational curve of order $n-s$ which is called the sth osculant of the R^n at t', or the osculant $(n-s)$-ic of the R^n at t'. Its parametric equations are, of course, simply (13). For instance, the osculant

conic of the R^3 at t' is

$$x_i = (a_i t' + b_i) t^2 + 2(b_i t' + c_i) t^2 + (c_i t' + d_i); \tag{14}$$

similarly, the osculant cubic of the R^4 at t' is

$$x_i = (a_i t' + b_i) t^3 + 3(b_i t' + c_i) t^2 + 3(c_i t' + d_i) t + (d_i t' + e_i); \tag{15}$$

and the osculant cubic of the R^5 at t' is

$$\begin{aligned} x_i = {} & (a_i t'^2 + 2b_i t' + c_i) t^3 + 3(b_i t'^2 + 2c_i t' + d_i) t^2 \\ & + 3(c_i t'^2 + 2d_i t' + e_i) t + (d_i t'^2 + 2e_i t' + f_i) \quad (i = 0, 1, 2). \end{aligned} \tag{16}$$

If the equation of the covariant line of (15) is formed, analogous to (10), this equation will involve t' to the second power, and may be arranged as a quadratic in t'. After these operations have been performed and t' changed to t, to indicate that it has become variable, the result may be written in the form

$$[\,|adx| - 3|bcx|\,]\, t^2 + [\,|aex| - 2|bdx|\,]\, t + [\,|bex| - 3|cdx|\,] = 0. \tag{17}$$

This is a covariant conic of the R^4, and if (17) is compared with (3), its parametric equations may be seen to be

$$\begin{aligned} \xi_i = {} & [(a_\mu d_\nu) - 3(b_\mu c_\nu)]\, t^2 + [(a_\mu e_\nu) - 2(b_\mu d_\nu)]\, t \\ & + [(b_\mu e_\nu) - 3(c_\mu d_\nu)] \quad (i, \mu, \nu = 0, 1, 2). \end{aligned} \tag{18}$$

It is readily seen that every R^n of even degree has a covariant conic whose parametric equations may be found in the same way; this conic for any such R^n of even degree is the envelope of the covariant lines of all osculant $(n-1)$-ics of the R^n.

If the equation of the covariant line of (16) is formed, the result will involve t' to the fourth power, and by a repetition of the process used above, the equations of the resulting covariant rational quartic in lines are

$$\begin{aligned} \xi_i = {} & [(a_\mu d_\nu) - 3(b_\mu c_\nu)]\, t^4 + [2(a_\mu e_\nu) - 4(b_\mu d_\nu)]\, t^3 \\ & + [(a_\mu f_\nu) + (b_\mu e_\nu) - 8(c_\mu d_\nu)]\, t^2 + [2(b_\mu f_\nu) - 4(c_\mu e_\nu)]\, t \\ & + [(c_\mu f_\nu) - 3(d_\mu e_\nu)] \quad (i, \mu, \nu = 0, 1, 2). \end{aligned} \tag{19}$$

In the same way the L_{n-2} of all osculant $(n-2)$-ics of R^n of odd order gives rise to a covariant rational quartic of the R^n. I shall treat this curve (19) in a later section of this paper as a type of covariant rational quartic of an R^n of odd order.

4. **Relation Between Transvectants and Envelopes of Covariant Lines of Osculants.** Equation (17) is exactly the same as (7) when $n = 4$ and $r = 3$. That is, the third transvectant of two line sections of the R^4 together with

the translation scheme is the same as the envelope of covariant lines of all osculant cubics of the R^4. Similarly, the curve whose equations are given in (19) is the third transvectant of two line sections of the R^5, or may be obtained as the envelope of the covariant lines of all osculant cubics of the R^5. In general, if n is even, the $(n-1)$th transvectant of two line sections of the R^n gives rise to a covariant conic of the R^n which may be defined as the envelope of the covariant lines of all osculant $(n-1)$-ics of the R^n. In the same way the $(n-3)$th transvectant of two line sections of the R^n yields a covariant rational sextic of the R^n which is identical with the envelope of covariant lines of all osculant $(n-3)$-ics of the R^n. More generally stated, the $(n-p)$th transvectant of two line sections of the R^n, when n is even, and p odd, is the same as the envelope of covariant lines of all osculant $(n-p)$-ics of the R^n. When n is odd the $(n-2)$th transvectant of two line sections gives rise to a rational quartic in lines which is a covariant of the R^n, and which may be looked upon as the envelope of covariant lines of all osculant $(n-2)$-ics of the R^n; the envelope of covariant lines of all osculant $(n-q)$-ics is a rational curve in lines of degree $2q$ of which the above quartic is a special case. A formal proof of these statements seems unnecessary as it would be evident to anyone who understands how to expand (7) and carry out the translation scheme suggested. I wish to consider in some detail the rational quartics mentioned above and in particular (19), which is the simplest of these with one exception, namely, if $n = 3$ this curve becomes the tangential equation of the R^3 which has only one invariant. Also, the R^5 is the most interesting plane curve in this connection for a reason which will appear later; it is what may be called self-conjugate.

5. **Invariants of R^n Illustrated by the Particular Case of the R^5.** An invariant of any R^n given by equations (2) is expressible in terms of three-rowed determinants of the type $|abc|$, or determinants formed by replacing the x's of (8) by any other subscribed letter, say by d's which would give $|abd|$. The invariants of (19) are invariants of the R^5; each three-rowed determinant of the matrix of coefficients of (19) is expressible as a quadratic function of the determinants of the type $|abc|$. In my dissertation* I showed how certain combinants could be interpreted as invariants of the R^4. These combinants form the complete set of invariants of the R^4, and are expressible in terms of eight quantities $\alpha, \alpha', \beta, \beta', \mu, \mu', \lambda, \lambda', \gamma$, and δ. I shall now give the values which must be assigned to these eight quantities so that they may be considered invariants of the R^5; these, of course, would not constitute the complete system of the R^5. Incidentally we may observe that the same process may be applied to find what values the greek letters must have so that these combinants mentioned may be interpreted as invariants of any R^n of odd order.

*These Transactions, vol. 12 (1911), pp. 295–310.

It is well known* that

$$(20)\qquad \begin{vmatrix} c_1f_2 - c_2f_1 & b_1e_2 - b_2e_1 & a_1d_2 - a_2d_1 \\ c_2f_0 - c_0f_2 & b_2e_0 - b_0e_2 & a_2d_0 - a_0d_2 \\ c_0f_1 - c_1f_0 & b_0e_1 - b_1e_0 & a_0d_1 - a_1d_0 \end{vmatrix} = |abd|\,|cef| - |ade|\,|bcf|.$$

Using this as a basis of calculation the values of the greek letters which must be substituted in Salmon's combinants† to obtain invariants of the R^5 are

$$\begin{aligned}(21)\qquad \alpha = &-16|cdf|\,|bcf| + 4|acf|\,|cef| + 4|cef|\,|bce| - 6|abf|\,|def| \\ &- 2|bcf|\,|bef| - 6|bde|\,|bef| + 48|bde|\,|cdf| - 48|bcd|\,|def| \\ &- 12|cde|\,|aef| - 96|cde|^2.\end{aligned}$$

$$\begin{aligned}(22)\qquad \beta = &\,2|aef|\,|acf| - 4|bdf|\,|acf| - 2|abe|\,|cef| - 4|bcf|\,|bde| \\ &+ 16|cdf|\,|ace| - 32|cdf|\,|bcd| - 6|aef|\,|ade| + 12|bdf|\,|ade| \\ &- 12|def|\,|abd| - 48|ade|\,|cde| + 12|bde|^2.\end{aligned}$$

$$(23)\qquad \lambda = 4|aef|\,|bcf| - 12|aef|\,|bde| - 12|abe|\,|def| + 8|ace|\,|cef| - 8|bcf|\,|bdf| + 24|bde|\,|bdf| - 16|cef|\,|bcd| - 48|cde|\,|bde|.$$

$$(24)\qquad \mu = 2|adf|\,|bcf| - 6|abd|\,|def| - 6|adf|\,|bde| + 4|acd|\,|cef| + 12|ade|\,|cde| - 6|bcf|^2 + 18|bde|\,|bcf| - 36|cde|\,|bce|.$$

$$\begin{aligned}(25)\qquad \gamma = &\,4|abf|\,|aef| - 8|ace|\,|aef| - 4|abe|\,|bef| - 32|ade|\,|bcf| \\ &+ 48|ace|\,|bdf| - 64|ace|\,|cde| - 8|abf|\,|bdf| - 16|abd|\,|cef| \\ &+ 16|bce|\,|bde| - 64|bdf|\,|bcd| + 128|bcd|\,|cde|.\end{aligned}$$

$$\begin{aligned}(26)\qquad \delta = &\,|acf|\,|adf| - 3|ade|\,|adf| - |abd|\,|cef| - 3|ade|\,|bde| \\ &+ 8|acd|\,|cdf| - 3|acf|\,|bcf| + 10|ade|\,|bcf| - 9|abc|\,|def| \\ &- 3|bcf|\,|bce| + 9|bce|\,|bde| - 72|cde|\,|bcd|.\end{aligned}$$

The values of α', β', λ', and μ' are not written out because they may be obtained by making the following interchanges in the values of the corresponding unprimed letters:

$$a \text{ and } f;\quad b \text{ and } e;\quad \text{and}\quad c \text{ and } d.$$

I have already explained what is meant by the self-conjugate property of the plane R^5 in a previous paper. It will be sufficient here with this reference

* Muir, *The Theory of Determinants*, pp. 142–143.

† *Higher Algebra*, third edition, pp. 200–206.

to recall that every invariant of the R^5 may be transformed either into itself or into another invariant of the R^5 by the substitution of complementary determinants and allowance for numerical coefficients. That is, if in Salmon's combinants A, B, C, and D the above values are substituted for the greek letters we have four invariants of the R^5; if in these invariants we substitute, for instance, $50|cef|$ for $|abd|$, and $5|aef|$ for $|bcd|$ and the other similar complementary determinants multiplied by a proper coefficient, we obtain four other invariants of the R^5. If the same operations are again performed multiples of the original invariants A, B, C, and D of the R^5 are formed. It is fairly evident upon reflection that this transformation is necessarily involutory. These eight invariants together with the cusp* and undulation invariants may be used as the basis of a study of invariants of the R^5; relations among them will be reserved for a later discussion.

PENNSYLVANIA STATE COLLEGE,
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*Annals of Mathematics, vol. 14 (1913), pp. 203–205.

A SET OF POSTULATES FOR GENERAL PROJECTIVE GEOMETRY*

BY

MEYER G. GABA

Since Klein promulgated his famous *Erlangen Programme*† it has been known that the various types of geometry are such that each is characterized by a group of transformations. In view of the importance of the concept of transformation in nearly all mathematics and perhaps especially in geometry, geometers may properly seek to develop the various types of geometry in terms of point and transformation. For euclidean geometry this has been done by Pieri.‡

This paper is devoted to a similar treatment of general projective geometry.§ One would naturally lay such postulates on the system of transformations so as to make the system form the group associated with the geometry. This was the scheme that Pieri used. His postulates make his transformations form the group of motions. In general projective geometry, however, this method is not necessary. If we are given the group of all projective transformations we can deduce the geometry from it but it will be shown in the sequel that we can also do that from a properly chosen semi-group belonging to that group.

Our basis, to repeat, is a class of undefined elements called points and a class of undefined functions on point to point‖ or transformations called collineations. For notation we will use small *Roman* letters to designate

* Read before the American Mathematical Society, April 26, 1913.

† F. Klein, *Vergleichende Betrachtungen über neuere geometrische Forschungen*, Erlangen, 1872. English translation by M. W. Haskell, Bulletin of the American Mathematical Society, vol. 2 (1893).

‡ M. Pieri, *Della geometria elementare come sistema ipotetico-deduttivo; monografia del punto e del mote*, Memorie delle Scienze di Torino (1899).

§ General projective geometry is defined by Veblen and Young as a geometry associated (analytically) with a general number field; that is, its theorems are valid, not alone in the ordinary real and the ordinary complex projective spaces but also in the ordinary rational spaces and in the finite spaces. This paper connects closely with the postulates for general projective geometry given by O. Veblen and J. W. Young in the American Journal of Mathematics, vol. 30 (1908), and in their *Projective Geometry*, Ginn and Company (1910).

‖ By point to point is meant that to every point p there corresponds a single point p' and no point p' is the correspondent of two distinct points p_1 and p_2.

points and small Greek letters for collineations. Thus

$$\tau(p_1, p_2, p_3) = p_1', p_2', p_3'$$

means that the collineation τ transforms the points p_1, p_2, p_3 into p_1', p_2', p_3' respectively. Line will be defined in terms of points and collineations. If we should interpret our undefined collineations as the group of all projective collineations, our defined line will satisfy the Veblen-Young assumptions for their undefined line and the postulates I to VIII that we will soon give are theorems in general projective geometry. This proves the consistency of our postulates. On the other hand, leaving collineation as undefined and using postulates I to VI* we are able to prove as theorems the Veblen-Young assumptions $A_1, A_2, A_3, E_0, E_1, E_2, \cdots, E_n, E_{n'}$, and P.† This shows that our six postulates are sufficient to establish the general projective geometry of n-dimensions. The undefined collineation will be proven to be a projective collineation which justifies the notation. If we desire that our class of collineations should be the group of all projective collineations we add postulate VII to the preceding six. To the Veblen-Young postulate H_0 corresponds our postulate VIII.

The independence of our assumptions is proven by the set of independence examples given at the end of this paper.

POSTULATE I. *There are at least $n + 2$ distinct points.*

POSTULATE II. *If τ_1 is a collineation and τ_2 is a collineation then the resultant of operating first with τ_1 and then with τ_2* (in notation $\tau_2 \tau_1$) *is also a collineation.*

DEFINITION. A linear set is a class of points such that:

(*a*) every collineation that leaves two distinct points of the class invariant leaves the class invariant,

(*b*) every collineation that leaves three distinct points of the class invariant leaves every point of the class invariant.

DEFINITION. Points belonging to the same linear set are called collinear.

DEFINITION. A linear set that contains at least three distinct points is called a line.‡

POSTULATE III. *If p_1, p_2, p_3 are three distinct collinear points and p_1', p_2', p_3' are three distinct collinear points then a collineation exists that transforms p_1, p_2, p_3 into p_1', p_2', p_3' respectively.*

POSTULATE IV. *If p_1, p_2, p_3, p_4 are four distinct points such that no three*

* Postulates I to VI explicitly require that our set of collineations form a semi-group but all sets of transformations that we found satisfying I–VI were groups. The question whether they necessarily form a group in the general case has not as yet been proven.

† The precise statement of these postulates is given later in this paper.

‡ It will be proved that every pair of distinct points is contained in one and only one line.

*are such that each is collinear with the same two distinct points then a collineation exists leaving p_1 and p_2 invariant and interchanging p_3 and p_4.**

DEFINITION. If p_1, p_2 are two distinct points, a line containing p_1 and p_2 or, in case no line exists containing p_1 and p_2, then the pair of points p_1, p_2 is called a one-space (P_1) containing p_1, p_2.

DEFINITION. If P_{k-1} is a $(k-1)$-space and p_0 is a point not contained in P_{k-1}, the class $P_k \equiv [P_{k-1}, p_0]$ of all points p collinear with the point p_0 and some point of P_{k-1} is called the k-space determined by P_{k-1} and p_0.

DEFINITION. k points are called independent of each other if there exists no $(k-2)$-space that contains them all.

POSTULATE V. *If $p_1, p_2, \cdots, p_{n+1}$ are $n+1$ distinct points of the same k-space, $k < n$, then a collineation distinct from the identity exists leaving $p_1, p_2, \cdots, p_{n+1}$ invariant.*†

POSTULATE VI. *If $p_1, p_2, \cdots, p_{n+2}$ are $n+2$ distinct points then there exists a k-space, $k \leq n$, that contains them all.*

POSTULATE VII. *If $p_1, p_2, \cdots, p_{n+2}$ are $n+2$ points of the same n-space such that every $n+1$ are independent, and $p_1', p_2', \cdots, p_{n+2}'$ are $n+2$ points of the same n-space such that every $n+1$ are independent then a collineation exists that transforms $p_1, p_2, \cdots, p_{n+2}$ into $p_1', p_2', \cdots, p_{n+2}'$ respectively.*

DEFINITION. A complete quadrangle is a figure consisting of four distinct coplanar points such that no three are collinear, called its vertices and six distinct lines containing the vertices in pairs called its sides. Two sides having no vertex in common are called opposite and points common to two opposite sides are called diagonal points.

POSTULATE VIII. *The diagonal points of a complete quadrangle are non-collinear.*

THEOREM 1. *If p_1 and p_2 are distinct points, there is not more than one line containing both p_1 and p_2.*

Let us assume that two lines P and P' exist such that each contains p_1 and p_2. If the two lines are distinct then at least one of the lines must contain a point not in the other. Let us assume that p_3' is a point of P' and not of P. Since P is a line it contains in addition to p_1 and p_2 a third point p_3. From Postulate III we know that a collineation τ exists such that $\tau(p_1, p_2, p_3) = p_1, p_2, p_3'$. But τ leaves two points of P invariant, therefore it leaves P invariant and p_3 cannot be transformed into p_3' which is a point not of P. Since we are led to a contradiction our assumption that p_1 and p_2 are contained in two distinct lines must be false.

THEOREM 2. *Two distinct lines cannot have more than one common point.*

* Compare Postulate IV with Theorem 4. Postulate IV is weaker than Theorem 4 as independence example — IV will show.

† Postulate V with Theorem 14 shows that all points lie in no P_k, $k < n$.

THEOREM 3. *If three points are such that each is collinear with the same two points, they are collinear and conversely.*

THEOREM 4. *If p_1, p_2, p_3, p_4 are four distinct points such that no three are collinear then a collineation τ exists such that $\tau(p_1, p_2, p_3, p_4) = p_1, p_2, p_4, p_3$.*

The theorem follows from Postulate IV and Theorem 3.

THEOREM 5. *If p_1, p_2 are two distinct points and q_1, q_2 are two distinct points then a collineation τ exists such that $\tau(p_1, p_2) = q_1, q_2$.*

If p_1, p_2 are on a line and q_1, q_2 are on a line, the theorem follows from Postulate III. If no three of the points p_1, p_2, q_1, q_2 are collinear and they are all distinct then by Theorem 4 we know that a collineation τ_1 exists such that $\tau_1(p_1, q_1, p_2, q_2) = p_1, q_1, q_2, p_2$ and a collineation τ_2 exists such that $\tau_2(p_1, q_1, q_2, p_2) = q_1, p_1, q_2, p_2$. Therefore by Postulate II a collineation $\tau_2\tau_1$ exists such that $\tau_2\tau_1(p_1, p_2) = q_1, q_2$.

There are, notation apart, two possible cases remaining which are: (1) q_2 collinear with p_1 and p_2 but q_1, q_2 on no line; (2) p_2 collinear with q_1 and q_2, but p_1, p_2 on no line. Let us first suppose that a point r exists such that r is non-collinear with every two of the points p_1, p_2, q_1, q_2. In the first case, collineations τ_3 and τ_4 exist such that $\tau_3(p_1, p_2, r, q_1) = q_1, p_2, r, p_1$ and $\tau_4(q_1, p_2, r, q_2) = q_1, q_2, r, p_2$, and therefore $\tau_4\tau_3$ is the required collineation. In the second case, collineations τ_5 and τ_6 exist such that $\tau_5(p_1, p_2, r, q_2) = p_1, q_2, r, p_2$ and $\tau_6(p_1, q_2, r, q_1) = q_1, q_2, r, p_1$ and in this case $\tau_6\tau_5$ is a collineation that transforms p_1, p_2 into q_1, q_2 respectively.

If no such point r exists then every point is contained in some one of the one-spaces determined by two of the four points p_1, p_2, q_1, q_2. Let us consider the case where p_1 and p_2 are on no line and where p_1 is collinear with $q_1\ q_2$ and let us further suppose that p_1 is distinct from q_1 and from q_2. All points are in the two-space or plane determined by the line $q_1\ q_2$ and the point p_2 is not in the line $q_1\ q_2$. If $n = 2$ then by Postulate V a collineation not the identity exists having p_1, q_1, q_2 as invariant points and there must therefore be at least one additional point r_1 on the one-space $q_1\ p_2$ or one additional point s_1 on the one-space $q_2\ p_2$. If r_1 exists then collineations τ_7 and τ_8 exist such that $\tau_7(p_1, p_2, r_1, q_2) = p_1, q_2, r_1, p_2$ and $\tau_8(q_1, q_2, p_1) = p_1, q_2, q_1$. If s_1 exists we have collineations τ_9 and τ_{10} such that $\tau_9(p_1, p_2, s_1, q_1) = p_1, q_1, s_1, p_2$ and

$$\tau_{10}(q_1, q_2, p_1) = q_2, p_1, q_1.$$

Hence $\tau_8\tau_7$ or $\tau_{10}\tau_9$ will be the required collineation according as r_1 or s_1 exists. If $n > 2$ there are by Postulate I at least $n + 2 > 4$ distinct points; hence the additional point r_1 or s_1 exists as before, and the argument is completed as in the case $n = 2$.

Let us now suppose that p_1 coincides with q_1. Since $n \geqq 2$, at least four

distinct points exist. Let us assume q_3, distinct from q_1 and q_2, exists on the one-space $q_1 q_2$. A collineation distinct from the identity exists leaving invariant q_1, q_2, q_3 if $n = 2$ and, if $n > 2$, q_1, q_2, q_3, and $n - 2$ other points; therefore a point r_1 distinct from q_2, p_2 must lie on the one-space $q_2 p_2$. If the existence of r_1 had been assumed, then we could have proven in a similar manner that q_3 existed and since one or the other must exist, both exist. Since the four points $p_1 = q_1, p_2, r_1, q_3$ are such that no three are collinear a collineation τ_{11} exists such that $\tau_{11}(p_1, p_2, r_1, q_3) = p_1, q_3, r_1, p_2$ and a collineation τ_{12} exists such that $\tau_{12}(p_1, q_2, q_3) = q_1, q_3, q_2$. The collineation $\tau_{12}\,\tau_{11}$ is the collineation that transforms $p_1 p_2$ into $q_1 q_2$. For all the other possible cases the proofs are very similar to the preceding and therefore need not be repeated.

THEOREM 6. *A line exists.*

If all linear sets contained but two points there would be but $k + 1$ points in a k-space. This would make Postulates I and VI contradictory and therefore at least one linear set contains more than two points and hence a line exists.

THEOREM 7. *Every collineation transforms lines into lines.*

Let τ_1 be any collineation that transforms the line $P = [p]$ into a set of points $Q = [q]$. We are to prove that the set of points Q constitute a linear set. Let τ_2 be any collineation that leaves two of the q's, say q_1 and q_2 invariant. The points q_1 and q_2 are the transforms under τ_1 of two points of P which we will call p_1 and p_2. By Theorem 5 there is a collineation τ_3 that transforms q_1, q_2 into p_1, p_2. Then $\tau_3 \tau_2 \tau_1(P) = P$ since the points p_1 and p_2 of P are left invariant. For the same reason $\tau_3 \tau_1(P) = P$. Therefore $\tau_3(Q) = P$, that is to say every p is the transform under τ_3 of some q and that for every q $\tau_3(q)$ is a p. But $\tau_3 \tau_2(Q) = P$ therefore $\tau_2(Q) = Q$. Hence any collineation that leaves two points of Q fixed leaves Q invariant.

Let τ_4 be any collineation that leaves three of the q's invariant, say q_1, q_2, q_3, where q_1, q_2, q_3 are the transforms under τ_1 of p_1, p_2, p_3 respectively. $\tau_3 \tau_4 \tau_1 [P] = P$ since p_1 and p_2 are left invariant. $\tau_3 \tau_4 \tau_1(p_1, p_2, p_3) = p_1, p_2, p_3'$. Since p_1, p_2, p_3 and p_1, p_2, p_3' are sets of collinear points a collineation τ_5 exists such that $\tau_5(p_1, p_2, p_3) = p_1, p_2, p_3$. Then

$$\tau_5 \tau_3 \tau_4 \tau_1(p_1, p_2, p_3) = p_1, p_2, p_3$$

and

$$\tau_5 \tau_3 \tau_1(p_1, p_2, p_3) = p_1, p_2, p_3.$$

Therefore the collineations $\tau_5 \tau_3 \tau_1$ and $\tau_5 \tau_3 \tau_4 \tau_1$ leave every point of P invariant. If τ_1 transforms p_i into q_i, $\tau_5 \tau_3$ must transform q_i into p_i and hence τ_4 must leave every point of Q invariant. We have shown that properties (a) and (b) of a linear set hold for Q and since P was a line (containing at least three points) Q is a line.

THEOREM 8. *If p_1 and p_2 are distinct points, there is at least one line containing both p_1 and p_2.*

We know that at least one line exists from Theorem 6. That line has two points q_1 and q_2. By Theorem 5 a collineation τ exists transforming q_1, q_2 into p_1, p_2 respectively. The line containing q_1, q_2 is transformed by τ into a line which contains p_1, p_2.

THEOREM 9. *Every collineation transforms a k-space into a k-space.*

THEOREM 10. *All points are not on the same line.*

THEOREM 11. *If p_1, p_2, p_3, p_4, p_5 are five distinct points such that p_1, p_2, p_3 are non-collinear, p_1, p_2, p_4 are collinear and p_1, p_3, p_5 are collinear, then there exists a point p_6 such that p_2, p_3, p_6 are collinear and p_4, p_5, p_6 are collinear.**

No three of the points p_2, p_3, p_4, p_5 are collinear for if they were p_1, p_2, p_3 would be collinear. By Theorem 4 a collineation τ exists such that

$$\tau(p_2, p_3, p_4, p_5) = p_2, p_4, p_3, p_5.$$

The collineation τ transforms the lines $p_3 p_5$ and $p_2 p_4$ into the lines $p_4 p_5$ and $p_2 p_3$ respectively. The point p_1 common to the lines $p_3 p_5$ and $p_2 p_4$ will therefore be transformed into a point p_6 common to the lines $p_4 p_5$ and $p_2 p_3$.

We have already proven as theorems the Veblen-Young postulates A_1, A_2, A_3, E_0, E_1, and E_2. The postulate A_1 is our Theorem 8; A_2 is our Theorem 1; A_3 (if p_1, p_2, p_3 are points not all on the same line and p_4 and p_5 ($p_4 \neq p_5$) are points such that p_1, p_2, p_4 are on a line and p_1, p_3, p_5 are on a line, there is a point p_6 such that p_2, p_3, p_6 are on a line and p_4, p_5, p_6 are on a line) is in content equivalent to Theorem 11; E_0 (there are at least three points on every line) is true from definition of line; E_1 (there exists at least one line) is Theorem 6; and E_2 is Theorem 9. We therefore know that our line satisfies the six preceding postulates that Veblen and Young lay down for their undefined line, hence all theorems that they derive from the six assumptions listed will hold in our geometry. One such theorem is:

THEOREM 12. *Let the k-space P_k be defined by the point p_0 and the $(k-1)$-space P_{k-1}, then*

(a) *There is a k-space on any $k+1$ independent points.*

(b) *Every line on two points of P_k has one point in common with P_{k-1} and is in P_k.*

(c) *Every P_g $(g < k)$ on $g+1$ independent points of P_k is in P_k.*

* Theorem 11 is essentially equivalent to the Veblen-Young postulate A_3: If p_1, p_2, p_3 are points not all on the same line and p_4 and p_5 ($p_4 \neq p_5$) are points such that p_1, p_2, p_4 are on a line and p_1, p_3, p_5 are on a line, there is a point p_6 such that p_2, p_3, p_6 are on a line and p_4, p_5, p_6 are on a line. The form of statement for Theorem 11 was suggested by Professor E. H. Moore as a substitute for A_3 since the latter is redundant in that it includes the obvious cases where p_4 is coincident with p_1 or p_2 or where p_5 is coincident with p_1 or p_3.

(*d*) *Every* P_g $(g < k)$ *on* $g + 1$ *independent points of* P_k *has a* P_{g-1} *in common with* P_{k-1} *provided all* $g + 1$ *points are not in* P_{k-1}.

(*e*) *Every line* P_1 *on two points of* P_k *has one point in common with every* P_{k-1} *in* P_k.

(*f*) *If* q_0 *and* Q_{k-1} (q_0 *not in* Q_{k-1}) *are any point and any* $(k-1)$*-space respectively of the* k*-space determined by* p_0 *and* P_{k-1}, *the latter space is the same as that determined by* q_0 *and* Q_{k-1}.

Another important theorem that Veblen and Young prove is:

THEOREM 13. *On* $k + 1$ *independent points there is one and but one* k*-space.*

THEOREM 14. *If* $k + 1$ *points of a* P_{k-1}, *such that every* k *are independent, are left invariant by a collineation* τ *then* τ *leaves every point of* P_{k-1} *invariant.*

This theorem clearly is true for a line or P_1. Let us assume that the theorem is true for a P_{g-1}. In a P_g if every $g + 1$ of $g + 2$ points are to be independent, then if $p_1, \cdots, p_g$ determine a P_{g-1}, p_{g+1} and p_{g+2} cannot lie in P_{g-1}, nor can the line $p_{g+1}\,p_{g+2}$ contain any of the points $p_1, \cdots, p_g$. The line $p_{g+1}\,p_{g+2}$ has a single point p_0 in common with P_{g-1} by Theorem 12 (*e*). When the $g + 2$ points are left invariant p_0, being the intersection of P_{g-1} and $p_{g+1}\,p_{g+2}$, is left invariant. The line P_{g-1} and the line $p_{g+1}\,p_{g+2}$ are each therefore left identically invariant. Let p be any point in P_g not in P_{g-1} nor on the line $p_{g+1}\,p_{g+2}$. The lines pp_{g+1} and pp_{g+2} each meet the P_{g-1} by Theorem 12 (*e*). When the given $g + 2$ points are left invariant, these lines and consequently their intersection is left invariant. The theorem being true for a g-space if true for a $(g-1)$-space and holding for a one-space is therefore true for a k-space.

THEOREM 15. *All points are not on the same* k*-space if* $k < n$.

It can easily be shown that every k-space has $k + 2$ points such that every $k + 1$ are independent. If these $k + 2$ points and $n - k - 1$ other points, which exist by Postulate I, are not in the k-space the theorem is true. If these $n + 1$ points are in the k-space then by Postulate V a collineation distinct from the identity exists leaving these $n + 1$ points invariant and therefore by Theorem 14 that collineation leaves the k-space identically invariant. Hence not all points can be in the k-space.

THEOREM 16. *There exist* $n + 2$ *points such that every* $n + 1$ *are independent.*

The definitions of perspectivity, projectivity, etc., can now be given exactly as Veblen and Young give them. We will now proceed to identify what we call a collineation with what they call a projective collineation. To do this we will first prove:

THEOREM 17. *Every central perspective correspondence between points of two lines can be secured by a collineation.*

Let the perspectivity be defined by p_1, p_2 having as their correspondents q_1, q_2. We have by Theorem 4, since no three of p_1, p_2, q_1, q_2 are collinear,

that the two collineations τ_1 and τ_2 exist such that

$$\tau_1(p_1, p_2, q_1, q_2) = p_1, q_2, q_1, p_2$$

and

$$\tau_2(p_1, q_2, q_1, p_2) = q_1, q_2, p_1, p_2,$$

therefore $\tau = \tau_2 \tau_1$ exists by Postulate II and is such that $\tau(p_1, p_2, q_1, q_2) = q_1, q_2, p_1, p_2$. The lines $p_1 q_1$ and $p_2 q_2$ will be left invariant by τ and so will their point of intersection o. The lines $p_1 p_2$ and $q_1 q_2$ are interchanged by τ as well as the lines $p_1 q_2$ and $p_2 q_1$. Let the intersection of $p_1 p_2$ and $q_1 q_2$ be r_3 and of $p_1 q_2$ and $p_2 q_1$ be r_4. The collineation τ must leave r_3 and r_4 invariant. There are two cases possible,* the first where o, r_3, and r_4 are non-collinear and the second where these three points are collinear. Let us first consider case 1. The line $r_3 r_4$ meets the lines $p_1 q_1$ and $p_2 q_2$ in points r_1 and r_2. The collineation τ leaves r_1, r_2, r_3, and r_4 invariant; hence every point of the line r_3, r_4 is left invariant by τ. Let p be any point on the line $p_1 p_2$ and call r the point of intersection of the line op with the line $r_1 r_2$. The line op is left invariant by τ since that collineation leaves two of its points, o and r, invariant. Since the line $p_1 p_2$ is transformed into the line $q_1 q_2$ by τ, p is transformed by τ into the intersection of the lines op and $q_1 q_2$ which we will call q.

For the second case, where o, r_3, and r_4 are collinear, the points o and r are coincident. The collineation $\tau^2 = \tau\tau$ leaves p_1, p_2, and r_3 invariant and therefore leaves every point of $p_1 p_2$ invariant. Hence if q denotes the point of $q_1 q_2$ into which τ transforms a point p of $p_1 p_2$ then τ must transform q into p. Hence the line pq is left invariant by τ. Since o, r_3, and r_4 are collinear the lines $p_1 q_1$, $p_2 q_2$, and $r_3 r_4$ are concurrent at o. If the line pq did not pass through o it would intersect the three lines $p_1 q_1$, $p_2 q_2$, and $r_3 r_4$ at three distinct points. But this would make the line pq identically invariant under the collineation τ and hence p would be invariant and coincide with q, but this is possible only if p is r_3 and we assumed that p was any point on the line $p_1 p_2$ therefore pq passes through o. Therefore the collineation τ makes correspond to the points of the line (p) the points of line (q) perspective to (p) with center of perspectivity o.

THEOREM 18. *If a projective correspondence exists between the points of two lines, then a collineation exists that transforms the points of the first line into the projectively corresponding points of the second line.*

A projective correspondence between the points of two lines is the resultant of a sequence of central perspectivities. By Theorem 16, each central perspective correspondence has associated with it a collineation. The resultant of the sequence of collineations corresponding to the sequence of perspectivities that define the projectivity is the required collineation.

* If Postulate VIII were assumed, the first case only would arise.

THEOREM 19. *If a projectivity leaves each of three distinct points of a line invariant it leaves every point of the line invariant.*

THEOREM 20. *If P is a P_n all points are in P.*

This theorem follows readily from Postulate VI and Theorem 12.

In addition to the Veblen-Young postulates A_1, A_2, A_3, E_0, and E_1 we have proven E_n, which is Theorem 15, E_n', which is Theorem 19, and P, which is Theorem 18. We have therefore proven all of the postulates that Veblen and Young assume for general projective geometry of n-space and consequently all the theorems that can be derived from their postulates hold in our geometry.

We can now prove

THEOREM 21. *Every collineation is a projective collineation.*

Let τ be any collineation. From Theorem 20 we know that there exist $n+2$ points such that every $n+1$ are independent. The collineation τ will transform these $n+2$ points which we will call $p_1, p_2, \cdots, p_{n+2}$ into $p_1', p_2', \cdots, p_{n+2}'$ respectively. There exists a projective collineation π (from the Veblen-Young geometry which is at our disposal) such that $\pi(p_1, p_2, \cdots, p_{n+2}) = p_1', p_2', \cdots, p_{n+2}'$. The points $p_1, p_2, \cdots, p_n$ determine an $(n-1)$-space which will meet the line $p_{n+1}\,p_{n+2}$ in a point p_0. Let p_0' denote the intersection of the $(n-1)$-space determined by $p_1', p_2', \cdots, p_n'$ with the line $p_{n+1}'\,p_{n+2}'$. This point, p_0', will correspond to p_0 both by transformation τ and transformation π. The line $p_{n+1}\,p_{n+2}$ having three of its points p_{n+1}, p_{n+2}, and p_0 transformed into p_{n+1}', p_{n+2}', and p_0 both by τ and π, will have every one of its points transformed into the same corresponding point of $p_{n+1}'\,p_{n+2}'$ both by τ and π. This is true for every line $p_i\,p_j$ $(i, j = 1, 2, 3, \cdots, n+2; i \neq j)$. Every plane $p_i p_j p_k$ will go into the plane $p_i'\,p_j'\,p_k'$ and every point in the first plane will have the same correspondent in the second plane both by τ and π. Let p_{ijk} be any point in the plane $p_i\,p_j\,p_k$. Draw the lines $p_i\,p_{ijk}$ and $p_j\,p_{ijk}$. These lines will intersect the lines $p_j\,p_k$ and $p_i\,p_k$ in points that we can call p_{jk} and p_{ik}. Transformations τ and π will transform p_i, p_j, p_{jk}, and p_{ik} into the same points, that is into p_i', p_j', p_{jk}', and p_{ik}'. To p_{ijk} will correspond the intersection of $p_i'\,p_{jk}'$ with $p_j'\,p_{ik}'$ by either τ or π. By continuing this process we can prove that to any point p there corresponds by either τ or π the same point p'. Hence τ and π are identical.

If we desire the set of collineations from which we start to be the group of all projective collineations we add Postulate VII to the postulates we have used and we have the theorem.

THEOREM 22. *Every projective collineation is a collineation.*

INDEPENDENCE EXAMPLES, $n \geq 3$

In the following a Roman numeral preceded by a minus sign denotes an example of a system in which the postulate denoted by that numeral is false but all the other postulates of the set I–VIII are true.*

– I. Let the class of points consist of a single element and the class of collineations of the identity transformation.

– II. Case 1, n odd. Let the class of points consist of $2(n+1)$ elements in $(n+1)/2$ sets of four. Let the collineations be all the transformations that permute the points of each set amongst themselves, the identity transformation excepted, together with all the transformations that permute the points of each of all but two sets amongst themselves but leaves one point of each of the two remaining sets invariant and transforms the other three points of each of these sets into the remaining three points of the other set. Each set is a line and there are no other lines.

Case 2, n even. Let the class of points consist of $2n+1$ points in $n/2$ sets of four and one single point. Let the collineations be transformations on the $n/2$ sets like those of Case 1 leaving the extra point invariant together with the transformations that permute the points of each of all but one of the sets amongst themselves, interchange the single point with one of the points of the remaining set and leaves none of the three other points of that set invariant. Each set is again a line and there are no other lines.

– III. Let the class of points be $2(n+1)$ elements in $n+1$ sets of two. The collineations are the 2^{n+1} transformations on the points leaving each pair invariant. The lines are the $n(n+1)/2$ tetrads of points each consisting of two pairs.

– IV. Case 1, n odd. Let the class of points consist of $2(n+1)$ elements in $(n+1)/2$ sets of four. Let the class of collineations be all the transformations that permute the points of each set or interchange the sets. Each set is a line.

Case 2, n even. Let the class of points consist of $2n+1$ elements in $n/2$ sets of four and one single point. Let the collineations be like those of case 1 on the sets and leave the extra point invariant. Each set is a line.

– V. Ordinary projective geometry of $(n-1)$-space.

– VI. Ordinary projective geometry of $(n+1)$-space.

– VII. Let the class of points be the class of all sets of $n+1$ rational numbers $(X_1, X_2, \cdots, X_{n+1})$, the set $(0, 0, \cdots, 0)$ excepted. The sets $(X_1, X_2, \cdots, X_{n+1})$ and $(KX_1, KX_2, \cdots, KX_{n+1})$ are understood to be equivalent for all rational values of k distinct from zero. Let the collineations be the class of all linear homogeneous transformations on $n+1$ variables,

*In – I, Postulates III–VIII are satisfied *vacuously*. The same is true of Postulates VII and VIII in – I and – III and of Postulate VII in – IV.

having rational coefficients and whose determinants of transformation are $(n+1)$th powers of rational numbers not zero.

– VIII. The finite projective geometry of n-space having three points on a line.*

* For finite projective geometries see O. Veblen and W. H. Bussey, *Finite projective geometries*, these Transactions, vol. 7 (1906), pp 241–259.

CERTAIN QUARTIC SURFACES BELONGING TO INFINITE DISCONTINUOUS CREMONIAN GROUPS*

BY

VIRGIL SNYDER AND F. R. SHARPE

1. **Statement of the problem.** The following paper has for its purpose the establishment of two theorems:

THEOREM I. *The quartic surface subjected to the single condition of passing through a non-hyperelliptic sextic curve of genus three is invariant under an infinite discontinuous group of birational transformations.*

THEOREM II. *The transformations of the infinite discontinuous group under which the most general quartic surface passing through a sextic curve of genus two remains invariant can be expressed in terms of cremonian transformations.*

In connection with the first theorem the equation of the surface is derived and the equations of the transformations are determined; it is shown that the transformations are cremonian and non-involutorial, and that no transformations exist other than those obtained. It is believed that this surface is the first illustration of one which possesses an infinite discontinuous group, but contains neither a pencil of elliptic curves, nor a net of hyperelliptic curves of genus two. The equation of the surface mentioned in the second theorem is found, and also the equations of two involutorial space transformations which generate the infinite group.

1. SURFACES THROUGH A SEXTIC OF GENUS THREE

2. **The cubic transformation.** Consider the birational transformations between the spaces (x) and (x') by means of the three bilinear equations†

$$A = \sum a_{ik}\, x_i\, x'_k = 0, \qquad B = \sum b_{ik}\, x_i\, x'_k = 0,$$
$$C = \sum c_{ik}\, x_i\, x'_k = 0 \tag{1}$$
$$(a_{ik} \neq a_{ki},\ b_{ik} \neq b_{ki},\ c_{ik} \neq c_{ki}).$$

* Presented to the Society, January 1, 1915.

† This transformation is discussed analytically by Doehelemann, *Geometrische Transformationen*, vol. 2, pp. 286–296, and synthetically by Sturm, *Theorie der geometrischen Verwandtschaften*, vol. 3, pp. 484–486 and vol. 4, pp. 370–371.

By solving these equations for x_k' in terms of x_1, x_2, x_3, x_4 we obtain

$$x_k' = \phi_k(x) \qquad (k = 1, 2, 3, 4). \tag{2}$$

The cubic surfaces $\phi_k(x) = 0$ all pass through the fundamental curve of the transformation (2). This curve is a space sextic C_6 of genus three, the most general of its kind. The ruled surface R_8 of order eight formed by the trisecants of C_6 is the fundamental surface of the transformation. It contains C_6 as a three-fold curve and no other multiple curve. Similarly, by solving the system (1) for x_i we obtain

$$x_i = \psi_i(x') \qquad (i = 1, 2, 3, 4). \tag{3}$$

Let C_6' and R_8' be respectively the fundamental curve and surface for the space (x').

3. **Equation of the surface.** The equation of the most general quartic surface passing through C_6 may be written in the form

$$F_4 = \sum d_{ik}\, x_i\, \phi_k(x) = 0 \qquad (d_{ik} \neq d_{ki}). \tag{4}$$

By means of equations (2) and (3) we may write the equation of the transformed surface in the form

$$F_4' = \sum d_{ik}\, \psi_i(x')\, x_k' = 0. \tag{5}$$

A general plane section of $F_4 = 0$ made by $\sum u_i x_i = 0$ goes into the residual section of $F_4' = 0$ with a cubic surface $\sum u_i \psi_i(x') = 0$ passing through C_6'. This residual section is also a space sextic $\overline{C}_6'$ of genus three. Hence $F_4' = 0$ contains ∞^3 coresidual space sextic curves $\overline{C}_6'$, images of the plane sections of $F_4 = 0$. Any two of the curves $\overline{C}_6'$ meet each other in four points. Similarly, the surface $F_4 = 0$ contains ∞^3 coresidual space sextic curves $\overline{C}_6$, images of the plane sections of $F_4' = 0$ under the transformation (3). Any two of the curves $\overline{C}_6$ meet in four points.

The points of C_6 are transformed into the generators of R_8'. The intersection of F_4' with R_8' consists of C_6' counted three times and of a space curve C_{14}' of genus three and order fourteen. Since C_6 and C_4 meet in six points, it follows that C_{14}' and $\overline{C}_6'$ meet in six points.

4. **A second cubic transformation.** Equation (4) may be written in the form

$$\begin{vmatrix} \sum a_{i1} x_i & \sum a_{i2} x_i & \sum a_{i3} x_i & \sum a_{i4} x_i \\ \sum b_{i1} x_i & \sum b_{i2} x_i & \sum b_{i3} x_i & \sum b_{i4} x_i \\ \sum c_{i1} x_i & \sum c_{i2} x_i & \sum c_{i3} x_i & \sum c_{i4} x_i \\ \sum d_{i1} x_i & \sum d_{i2} x_i & \sum d_{i3} x_i & \sum d_{i4} x_i \end{vmatrix} = 0. \tag{6}$$

If we consider the fourth bilinear equation $D = 0$ we see from (6) that $F_4 = 0$ is also transformed into $F_4' = 0$ by the transformation defined by $A = 0$, $B = 0$, $D = 0$, and by two other similar ones. These transformations are distinct for an arbitrary point of space but, from a known property of determinants,* they are identical for points of $F_4 = 0$.

From the form of equation (6) it is seen that $F_4 = 0$ is also transformed into $F_4' = 0$ by means of any three of the four bilinear equations

$$(7) \quad (\sum_i a_{ik} x_i) x_1' + (\sum_i b_{ik} x_i) x_2' + (\sum_i c_{ik} x_i) x_3' + (\sum_i d_{ik} x_i) x_4' = 0 \qquad (k = 1, 2, 3, 4).$$

The four transformations thus obtained are distinct for an arbitrary point of space, but are identical for points of $F_4 = 0$.

We shall designate the transformation defined by (2) by T_1, and its inverse (3) by T_1^{-1}. Similarly, that defined by (7) by T_2, and its inverse by T_2^{-1}. By means of T_2 the planes of (x) are transformed into cubic surfaces of (x') which have a fundamental curve of order six and of genus three in common. The surface $\psi_1(x') = 0$ is the image of $x_1 = 0$ by both transformations, which have different fundamental curves, namely, the two sextics which together form the complete intersection of $\psi_1(x') = 0$ with $F_4' = 0$. Hence we see that C_6' belongs to a triply infinite system of coresidual curves, any two of which have four points in common. Similarly for the systems on $F_4 = 0$.

5. **An infinite discontinuous group.** The product of the two transformations T_1, T_2^{-1} leaves $F_4 = 0$ invariant. By T_1 a plane section is transformed into a space sextic of (x'), which by T_2^{-1} is transformed into a C_{14} of (x). Since the two transformations do not have common fundamental elements, the product $T_1 T_2^{-1}$ is not periodic and therefore generates an infinite group.

6. **Linear systems of curves on $F_4 = 0$.** If we apply the method used by Severi in his study of the quartic surface through a sextic curve of genus two† we may choose the plane sections C_4 and one system of sextics C_6 for a minimum basis. The other system of sextics has the symbol $3C_4 - C_6$. Any system of curves on the surface is expressible in the form $\lambda C_4 + \mu C_6$, in which λ and μ are integers. If the curve is of grade n and of genus π, we have the formula

$$n = 2\pi - 2 = 4\lambda^2 + 12\lambda\mu + 4\mu^2.$$

The surface therefore contains no system of curves of even genus, nor pencil of elliptic curves.

7. **Possible birational transformations.** Suppose a birational transforma-

* See, *e. g.*, Burnside and Panton, *Theory of Equations*, 2d edition, p. 266.

† F. Severi, *Complementi alla teoria della base per la totalità delle curve di una superficie algebrica*, Rendiconti del Circolo Matematico di Palermo, vol. 30 (1911), pp. 265–288.

tion exists which leaves $F_4 = 0$ invariant. Let it change C_4 into $\alpha C_4 + \beta C_6$ and C_6 into $\gamma C_4 + \delta C_6$, where $\alpha\delta - \beta\gamma = \pm 1$. Since $[C_4, C_4] = 4$, $[C_4, C_6] = 6$, $[C_6, C_6] = 4$, we must have the relations

$$\alpha^2 + 3\alpha\beta + \beta^2 = 1, \qquad \gamma^2 + 3\gamma\delta + \delta^2 = 1,$$

$$2\alpha\gamma + 3(\alpha\delta + \beta\gamma) + 2\beta\delta = 3.$$

If $\alpha\delta - \beta\gamma = 1$, we may write

$$2\alpha = t + 3u, \qquad \beta = -u, \qquad \gamma = u, \qquad 2\delta = t - 3u,$$

so that

$$t^2 - 5u^2 = 4.$$

Put $u = 1$, so that $t = 3$. In this case we have

$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \begin{vmatrix} 3 & -1 \\ 1 & 0 \end{vmatrix} = \tau.$$

It is seen that any power of τ may be expressed by the recurring formula

$$\tau^p = \begin{vmatrix} g_p & -g_{p-1} \\ g_{p-1} & -g_{p-2} \end{vmatrix},$$

in which $g_p = 3g_{p-1} - g_{p-2}$, with $g_1 = 3$, $g_0 = 1$, $g_{-1} = 0$.

If

$$\alpha\delta - \beta\gamma = -1,$$

one solution is $\alpha = 0, \beta = 1, \gamma = 1, \delta = 0$. Set

$$\sigma = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}.$$

The product $\tau^p \sigma$ is also an involution, and all transformations of determinant -1 are of the form $\tau^p \sigma$. Hence we have the theorem:

THEOREM. *All the birational transformations which exist and leave the surface $F_4 = 0$ invariant are of the form τ^p or $\tau^p \sigma$. The former are not periodic and the latter are involutorial.*

The transformation $T_2 T_1^{-1}$ is readily seen to be τ^2.

8. **Non-existence of involutions on $F_4 = 0$.** If σ or $\tau^2 \sigma$ exists on $F_4 = 0$, there must be a transformation L which transforms $F_4 = 0$ into $F_4' = 0$ such that

$$T_2 L^{-1} = \tau^2 \sigma, \qquad T_1 L^{-1} = \sigma.$$

From the definition of T_1 and T_2 it is easily seen that L must be linear. But a simple illustration shows that $F_4 = 0$ and $F_4' = 0$ are not linearly equivalent.

If

$$A = x_1 x_2' + x_2 x_3' = 0, \qquad B = -x_1 x_3' + x_2 x_4' - x_3 x_2' = 0,$$

$$C = x_2 x_1' - x_4 x_4' = 0, \qquad D = -x_1 x_3' + x_3 x_1' = 0,$$

the equation of $F_4 = 0$ is

$$x_1^2 x_2^2 - x_1^2 x_3 x_4 + x_2 x_3^2 x_4 = 0$$

and of $F_4' = 0$ is

$$x_4' (x_1' x_2' x_4' - x_2' x_3'^2 + x_1' x_3'^2) = 0.$$

These surfaces are evidently not linearly equivalent.

The transformation σ, which should interchange the plane sections and the sextics of one family, is thus seen not to exist. Similarly, the operation τ, which should change the plane sections into one system of sextics and the other system of sextics into the plane sections, does not exist. Hence the only effective transformations are powers of τ^2.

2. The general quartic surface through a sextic curve of genus two*

9. **Cubic surfaces through a sextic curve of genus two.** A space sextic curve C_6 of genus two has one quadrisecant s_4. A plane through s_4 cuts C_6 in two residual points. The locus of the lines joining these residual points as the plane turns about s_4 is a cubic ruled surface $f_3 = 0$ having s_4 for double directrix. Through C_6 pass two other linearly independent cubic surfaces $f_1 = 0$ and $f_2 = 0$. Since s_4 lies on $f_1 = 0$ and $f_2 = 0$, their residual intersection is a conic. The equations of the conic may be taken as

$$x_3 = 0, \qquad Q_3(x_1, x_2, x_4) = 0,$$

in which $Q_3 = 0$ is a quadric cone with vertex at $(0, 0, 1, 0)$. If s_4 has the equations $x_1 = 0, x_2 = 0$, and if $Q_1 = 0, Q_2 = 0$ are two quadrics through s_4, we may write

$$f_1 \equiv x_2 Q_3 - x_3 Q_2 = 0, \qquad f_2 \equiv x_3 Q_1 - x_1 Q_3 = 0.$$

The equation of the cubic ruled surface then has the form

$$f_3 \equiv x_1 Q_2 - x_2 Q_1 = 0.$$

Now consider the conic $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0, \lambda_1 Q_1 + \lambda_2 Q_2 + \lambda_3 Q_3 = 0$. Through it pass the pencil of cubic surfaces $a_1 f_1 + a_2 f_2 + a_3 f_3 = 0$, where $a_1 \lambda_1 + a_2 \lambda_2 + a_3 \lambda_3 = 0$. Hence in each plane through $(0, 0, 0, 1)$ lies a conic through which pass a pencil of cubic surfaces, the residual intersection being C_6 and s_4.

* G. Fano, *Sopra alcune superficie del quarto ordine rappresentabili sul piano doppio*, Rendiconti del Reale Istituto Lombardo, vol. 39 (1906), pp. 1071–1086.

Let the equations of the simple directrix of $f_3 = 0$ be $x_3 = 0$, $x_4 = 0$; let $x_4 = 0$ meet s_4 in a point of C_6, and let the two generators of $f_3 = 0$ in this plane be $x_1 = 0$ and $x_2 = 0$. The equation of $f_3 = 0$ now has the form

$$f_3 = (x_1^2 + kx_1x_2 + x_2^2)x_4 - x_1x_2x_3 = 0. \tag{8}$$

The projecting cone K_5 of C_6 from $(0, 0, 1, 0)$ is of order five, has s_4 for triple line, one generator, say $x_2 = 0$, $x_4 = 0$ for double line, and the other, $x_1 = 0$, $x_4 = 0$ for simple line. Its equation may be written in the form

$$\begin{aligned} &x_4^2\{(Px_1 + Sx_2)(x_1^2 + kx_1x_2 + x_2^2) + Rx_1x_2^2\} \\ &+ x_4x_2\{(px_1 + sx_2)(x_1^2 + kx_1x_2 + x_2^2) + (qx_1 + rx_2)x_1x_2\} \\ &+ x_1x_2^2(ax_1^2 + bx_1x_2 + cx_2^2) = 0. \end{aligned} \tag{9}$$

Since C_6 and s_4 lie on $K_5 = 0$ and on $f_3 = 0$, by combining (8) and (9) we have for the equations of the other cubic surfaces,

$$\begin{aligned} f_1 &= x_3\{x_4(Px_2 + Sx_2) + x_2(px_1 + sx_2)\} \\ &\quad + x_2\{Rx_4^2 + x_4(qx_1 + rx_2) + x_2(ax_1^2 + bx_1x_2 + cx_2^2)\} = 0, \\ f_2 &= x_3\{(S - kP)x_1x_4 - Px_2x_4 + Px_1x_3 + x_1(px_1 + sx_2)\} \\ &\quad - x_1\{Rx_4^2 + x_4(qx_1 + rx_2) + x_2(ax_1^2 + bx_1x_2 + cx_2^2)\} = 0. \end{aligned} \tag{10}$$

10. **Quartic surfaces through C_6.** The equation $\sum m_{ik}x_if_k = 0$ represents a quartic surface passing through c_6, but it also passes through s_4. The equation contains but eleven independent homogeneous constants, since $x_1f_1 + x_2f_2 + x_3f_3$ vanishes identically. If $F = 0$ is the equation of a quartic surface through C_6 but not through s_4, the equation

$$\phi_4 = F + \sum m_{ik}x_if_k = 0$$

represents a surface passing through C_6 and containing eleven non-homogeneous constants; it is therefore the most general quartic surface through C_6.

We may take for $F = 0$ a quartic having $x_3 = 0$, $x_4 = 0$ for double line. The intersection of $F = 0$ and $f_3 = 0$ is C_6 and a residual sextic $\bar{C}_6$ of genus two, having s_4 for quadrisecant, and meeting C_6 in twelve other points. Since any quartic surface through C_6 but not through s_4 will suffice, we may think of the residual $\bar{C}_6$ as composite and as projected from $(0, 0, 1, 0)$ by the composite quintic cone

$$x_4^2\{(Sx_1 + Px_2)(x_1^2 + kx_1x_2 + x_2^2) + Rx_1^2x_2\} = 0. \tag{11}$$

To obtain the equation of $F = 0$ we write the general equation of a quartic passing through $x_3 = 0$, $x_4 = 0$ twice, and determine the coefficients from the condition that, if x_3 is eliminated by means of (8), the resultant shall be

composite, having for factors the first members of (2) and (11). Hence we find

$$F = R^2 x_4^4 + R(2S - kP) x_3 x_4^3 + (PR + P^2 - kPS + S^2) x_3^2 x_4^2$$
$$+ R(qx_1 + rx_2) x_4^3 + \{R(px_1 + sx_2) + S(qx_1 + rx_2) - Prx_1$$
$$+ P(q - kr) x_2\} x_3 x_4^2 + \{S(px_1 + sx_2) - Psx_1$$
$$+ (Pr + Pp - ksP) x_2\} x_3^2 x_4 + Psx_2 x_3^3 + R(ax_1^2 + bx_1 x_2 + cx_2^2) x_4^2$$
$$+ \{S(ax_1^2 + bx_1 x_2 + cx_2^2) + P(a - c) x_1 x_2 + P(b - kc) x_2^2\} x_3 x_4$$
$$+ Pcx_2^2 x_3^2 = 0.$$

Through the composite $\overline{C}_6$ determined by (8) and (11) pass the cubic surfaces

$$f_1' \equiv x_2 Q_3' - x_3 Q_2' = x_2 Rx_4^2 - x_3\{(S - kP) x_2 x_4 - Px_1 x_4 + Px_2 x_3\} = 0,$$
$$f_2' \equiv x_3 Q_1' - x_1 Q_3' = -x_1 Rx_4^2 - x_3\{Sx_1 x_4 + Px_2 x_4\} = 0.$$

The curves C_6 and $\overline{C}_6$ meet the plane quartic curve

$$F = 0, \qquad \sum_{i=1}^{3} \lambda_i x_i \equiv (\lambda x) = 0,$$

each in six points, lying respectively on the conics $(\lambda x) = 0$, $(\lambda Q) = 0$, and $(\lambda x) = 0$, $(\lambda Q') = 0$. This set of twelve points is the complete intersection of C_4 and $f_3 = 0$. The curve of intersection of $F = 0$ and $(\lambda x) = 0$ therefore lies on the quartic surface

$$(\lambda Q)(\lambda Q') + f_3 \sum_{i=1}^{4} A_i x_i = 0.$$

By making this equation simultaneous with $(\lambda x) = 0$, we find

$$A_1 = \lambda_1 \lambda_3 a, \qquad A_2 = \lambda_2 \lambda_3 b - \lambda_1 \lambda_3 c, \qquad A_3 = \lambda_1 \lambda_3 p - \lambda_1 \lambda_3 s - \lambda_1 \lambda_2 P,$$
$$A_4 = \lambda_2 \lambda_3 q - \lambda_1 \lambda_3 r - (\lambda_1^2 - k\lambda_1 \lambda_2 + \lambda_2^2) P.$$

For the general quartic surface $\phi_4 = F + \sum m_{ik} x_i f_k = 0$ we make use of the fact that the intersection of $\sum m_{ik} x_i f_k = 0$ and $(\lambda x) = 0$ lies on the quartic surface

$$(\lambda Q)(x_2 \textstyle\sum m_{ik} x_i - x_1 \sum m_{i2} x_i) + f_3(\lambda_1 \sum m_{i1} x_i + \lambda_2 \sum m_{i2} x_i$$
$$+ \lambda_3 \textstyle\sum m_{i3} x_i) = 0.$$

We shall say that $(\lambda Q')$ becomes $(\lambda Q'')$ and that $\sum A_i x_i$ goes into $\sum A_i' x_i$.

11. **Involutorial transformations of $\phi_4 = 0$.** In every plane through $(0, 0, 0, 1)$ the quartic surface determines a line which meets each of two conics in two points. These four points lie on the given quartic surface.

Given a point P on $\phi_4 = 0$. By substituting its coördinates in the equations $(\lambda x) = 0$, $(\lambda Q) = 0$, values of $\lambda_1 : \lambda_2 : \lambda_3$ may be determined. These values are to be substituted in the equation $\sum A_i' x_i = 0$. The two equations $(\lambda x) = 0$, $\sum A_i' x_i = 0$ determine the line. It meets $(\lambda Q) = 0$ in two points, one of which is P; hence the coördinates of the other are rational functions of those of P. This operation defines an involutorial transformation I_1 under which the surface $\varphi_4 = 0$ is invariant. If we substitute the coördinates of P in $(\lambda Q'') = 0$ instead of in $(\lambda Q) = 0$, and proceed in the same way, we obtain a second involution I_2.

12. **Depiction on a double plane.** If $\lambda_1, \lambda_2, \lambda_3$ are taken as the coördinates of a point K in a plane then K corresponds to both P and to the image P_1 in I_1, hence the surface $\varphi_4 = 0$ is depicted on a double plane. When P and P_1 coincide, the line $(\lambda x) = 0$, $\sum A_i' x_i = 0$ is tangent to the quadric $(\lambda Q) = 0$. If we eliminate the coördinates x_i from these equations and the equation which expresses the condition for tangency, we obtain the equation of the sextic curve of branch points on the double plane.*

13. **Involutions expressed by cremonian transformations.** When the point $P \equiv (y_1, y_2, y_3, y_4)$ is not on the surface $\varphi_4 = 0$, consider the pencil of quartic surfaces $l_1 \phi_4 + l_2 f_3 \sum u_i x_i = 0$, and choose $l_1 : l_2$ so that the surface passes through P. The equation $\sum A_i' x_i = 0$ changes to

$$l_1 \sum A_i x_i + l_2 \lambda_3^2 \sum u_i x_i = 0,$$

while Q_i, Q_i'' remain unchanged.

If $P_1 \equiv (y_1', y_2', y_3', y_4')$ we have

$$\frac{\sum A_i' y_i'}{\sum u_i y_i'} = \frac{\sum A_i' y_i}{\sum u_i y_i} = \frac{-l_2}{l_1},$$

$$(\lambda y) = 0, \qquad (\lambda Q''(y)) = 0, \qquad (\lambda y') = 0, \qquad (\lambda Q''(y')) = 0.$$

If x_3 and x_4 are eliminated from $(\lambda x) = 0$, $l_1 \sum A_i' x_i + \lambda_3^2 l_2 \sum u_i x_i = 0$, and $(\lambda Q) = 0$, we obtain an equation of the form

$$H_{11} x_1^2 + 2H_{12} x_1 x_2 + H_{22} x_2^2 = 0,$$

in which each coefficient is a rational function of $\lambda_1, \lambda_2, \lambda_3, l_1, l_2$. But

$$y_1 y_1' / y_2 y_2' = H_{22}/H_{11},$$

and hence

$$y_1 = \sigma y_2' \lambda_3 H_{22}, \qquad y_2 = \sigma y_1' \lambda_3 H_{11},$$
$$y_3 = -\sigma (\lambda_1 H_{22} y_2' + \lambda_2 H_{21} y_1'), \tag{12}$$

from which y_4 can be found in terms of (y') by means of

$$l_1 \sum A_i y_i + l_2 \lambda_3^2 \sum u_i y_i = 0.$$

* See Fano, loc. cit., p. 1071.

Since $l_1, l_2, \lambda_1, \lambda_2, \lambda_3$ have exactly the same values when expressed in terms of (y) as of (y'), therefore equations (12) have also the same form whether solved for y_i in terms of (y'), or for y_i' in terms of (y), and are applicable for every point of space. Similarly for the involution I_2. Hence we have the theorem:

THEOREM. *The involutions I_1, I_2, belonging to the quartic surface through a sextic curve of genus two, can be expressed in terms of cremonian transformations which are birational and involutorial for all space.*

CORNELL UNIVERSITY,
August, 1914.

THE FUNCTIONS OF A COMPLEX VARIABLE DEFINED BY AN ORDINARY DIFFERENTIAL EQUATION OF THE FIRST ORDER AND THE FIRST DEGREE*

BY

JOSEPH SLEPIAN

Beginning with Cauchy's† fundamental theorem in 1820, much has been done toward answering the two fundamental questions: Do there exist solutions of a given differential equation or system of equations, which satisfy given initial conditions, and how far are such solutions determined by these initial conditions?

The case in which the coefficients of the system are analytic throughout the domain of the independent variable was in the main disposed of by Cauchy and his contemporaries, but the cases in which the coefficients present singularities, has continued to occupy the attention of mathematicians up to the present time. Almost all of these researches, however, are concerned with the existence and nature of a solution only in a certain restricted domain for the independent variables.

In the present paper we propose to study the ordinary equation of the first order and first degree, $dy/dx = P(x, y)/Q(x, y)$, where $P(x, y)$ and $Q(x, y)$ are polynomials in the two complex variables x and y, for all values of x and y. The first part of the paper is given over to a statement of the well-known existence theorems required later. The functions defined by such an equation will present no other singularities than branch points or poles, except possibly at a finite number of points which may be determined algebraically, and these solutions are in general infinitely many valued functions over the x-plane.

These functions have been largely investigated by Boutroux‡ who studies the manner of interchange of their branches and the grouping of their branch

* Presented to the Society, April 26, 1913.

† Picard, *Traité d'analyse*, vol. 2, chap. 11.

‡ Rendiconti del Circolo Matematico di Palermo, vol. 24 (1907), pp. 209–222. Journal de mathématiques, ser. 6, vol. 6 (1910), pp. 137–199. *Leçons sur les fonctions définies par les équations différentielles du premier ordre.* Gauthier-Villars, Paris, 1908.

points in the neighborhood of the fixed singularities, and the manner of growth ("croissance") of single branches of these functions over extended regions in the x-plane.

In this paper we shall not make a study of the individual integrals of an equation, but rather of the "im grossen" properties of the totality of all the integrals of the equation. We shall be guided by the well-known facts in the case of real variables, due largely to Poincaré.* In this case the solutions may be represented by a family of curves in the x,y-plane; there are a finite number of singular points through which more than one curve may pass, and there are certain closed limit curves or cycles about which other curves wind spirally.

The second part of the present paper shows that the closed cycle has a close analogy in the complex case; here every solution of the equation has *limit solutions* which, with respect to this integral, play a part analogous to that of the closed cycle in the real case. These limit solutions are in general infinite in number, whereas in the real case the limiting closed cycles are finite in number. However, by applying a sieve-like process due to Birkhoff,† we are led to certain *cyclic* systems of limit solutions which have notable properties. There is in general only one such system, which probably coincides *in general* with the totality of solutions, but which does not always do so.

The last three parts of this paper deal with the interrelation of the fixed singularities, limit solutions, and cyclic systems of limit solutions. In particular, it is shown that, if all the fixed singularities of the equation are of the simplest type, any two integrals pass through a common fixed singularity or have a limit solution in common. The case when algebraic solutions exist is especially considered.

I wish here to express my thanks to Professor G. D. Birkhoff for his kind assistance and many suggestions in preparing this paper.

1. Preliminary theorems

The principal existence theorem in the theory of differential equations is that of Cauchy.

Theorem of Cauchy. *Given the differential equation, $dy/dx = f(x, y)$. If $f(x, y)$ is analytic in the region, $|x - x_0| \leqq r$, $|y - y_0| \leqq \rho$ then there exists one and only one function of x, $y(x)$, which satisfies the given equation, which at the point x_0 takes on the value y_0, and which is analytic within a circle about x_0, whose radius is not less than*

$$r(1 - e^{-\rho/(2Mr)})$$

*Journal de mathématiques, ser. 3, vol. 7 (1881), p. 375 and vol. 8 (1882), p. 251.

†Bulletin de la Société mathématique de France, vol. 40 (1912), pp. 305–323.

where M is the maximum absolute value of $f(x, y)$ in the region

$$|x - x_0| \leqq r, \qquad |y - y_0| \leqq \rho.$$

Throughout this paper, as in this theorem, by the term "a circle about a point," we shall mean a circle whose center is at that point.

Picard* has further proved that if a function which satisfies the differential equation can be extended analytically along a path L to x_0 (but perhaps not beyond), and if this function when taken along L to x_0 approaches the value y_0, then this function must be identical with the function $y(x)$ of the theorem. This result shows that the function under these conditions can be extended past the point x_0.

For the "im grossen" theory of the first order differential equation,

$$dy/dx = P(x, y)/Q(x, y)$$

where P and Q are polynomials without common factors, we need to consider the point at infinity in both the x- and the y-planes.

If we make the transformation, $\bar{y} = 1/y$ we obtain a new equation of the same form, $d\bar{y}/dx = P_2(x, \bar{y})/Q_2(x, \bar{y})$, whose integrals are the reciprocals of the integrals of the given equation. If $P_2(x, y)/Q_2(x, y)$ is analytic at $x = x_0$, $y = 0$, then by the theorem of Cauchy there is a function $\bar{y}(x)$, which satisfies the equation $d\bar{y}/dx = P_2(x, \bar{y})/Q_2(x, \bar{y})$ and which is analytic and vanishes at x_0. If $\bar{y}(x)$ does not vanish identically, then there is one, and only one, function of x, $y(x)$ which satisfies the equation $dy/dx = P(x, y)/Q(x, y)$ and which at the point x_0 has a pole. If the function $\bar{y}(x)$ vanishes identically then the equation $dy/dx = P(x, y)/Q(x, y)$ has no integral which at x_0 becomes infinite. In this case, to secure uniformity of statement in the theorems of this paper, we shall say that the equation admits the *ideal* integral $y = \infty$.

To treat the case where x is infinite, we make the transformation, $\bar{x} = 1/x$, obtaining the equation, $dy/d\bar{x} = P_1(\bar{x}, y)/Q_1(\bar{x}, y)$; and, to treat the case where both x and y are infinite, we make the transformation, $\bar{x} = 1/x, \bar{y} = 1/y$, obtaining $d\bar{y}/d\bar{x} = P_3(\bar{x}, \bar{y})/Q_3(\bar{x}, \bar{y})$. We obtain results for these cases entirely similar to the one stated for the case where y becomes infinite.

Thus, if the point at infinity is treated after the manner indicated, it is found to play a part no different from a finite point in the x- or y-plane. Throughout this paper, even if no special mention of the fact is made, the possibility of a variable becoming infinite is not to be excluded.

We have next to consider the points at which $P(x, y)/Q(x, y)$ (or the corresponding quotient P_i/Q_i) fails to be analytic. These points have been divided into two categories, the movable singularities, and the fixed or essen-

* Picard, *Traité d'analyse*, vol. 2, chap. 11.

tial singularities. The first category is made up of these cases:

(a): x_0, y_0 finite, $\quad Q(x_0, y_0) = 0, \quad Q(x_0, y) \not\equiv 0, \quad P(x_0, y_0) \neq 0;$

(b): x_0 infinite, $\quad Q_1(0, y_0) = 0, \quad Q_1(0, y) \not\equiv 0, \quad P_1(0, y_0) \neq 0;$

(c): y_0 infinite, $\quad Q_2(x_0, 0) = 0, \quad Q_2(x_0, y) \not\equiv 0, \quad P_2(x_0, 0) \neq 0;$

(d): x_0, y_0 infinite, $\quad Q_3(0, 0) = 0, \quad Q_3(0, y) \not\equiv 0, \quad P_3(0, 0) \neq 0.$

The following theorem is characteristic of these cases.

THEOREM. *If* $Q(x_0, y_0) = 0$, $Q(x_0, y) \not\equiv 0$, and $P(x_0, y_0) \neq 0$, *then there is one and only one function of* x, $y(x)$ *which satisfies the differential equation,* $dy/dx = P(x, y)/Q(x, y)$, *takes on the value* y_0 *at* x_0 *and has at* x_0 *a simple branch point.*

Here also the solution is unique in the sense defined above.

The other category of singularities of $P(x, y)/Q(x, y)$, the fixed singularities, we find it convenient to further subdivide into two classes, denoting them respectively as fixed singularities of the first kind, and of the second kind. The fixed singularities of the first kind are the points x_0, y_0 as follows:

(a): x_0, y_0 finite, $\quad P(x_0, y_0) = 0, \quad Q(x_0, y_0) = 0, \quad Q(x_0, y) \not\equiv 0;$

(b): x_0 infinite, $\quad P_1(0, y_0) = 0, \quad Q_1(0, y_0) = 0, \quad Q_1(0, y) \not\equiv 0;$

(c): y_0 infinite, $\quad P_2(x_0, 0) = 0, \quad Q_2(x_0, 0) = 0, \quad Q_2(x_0, y) \not\equiv 0;$

(d): x_0, y_0 infinite, $\quad P_3(0, 0) = 0, \quad Q_3(0, 0) = 0, \quad Q_3(0, y) \not\equiv 0.$

A singularity of this kind has evidently two coördinates, x_0, y_0, so that at x_0 only those integrals which there take on the value y_0 fail to come under the preceding theorems.

The fixed singularities of the second kind include the only remaining singularities, i. e., the cases where $Q(x_0, y) \equiv 0$, $Q_1(0, y) \equiv 0$, $Q_2(x_0, y) \equiv 0$, or $Q_3(0, y) \equiv 0$, if any of these cases occur. A singularity of this kind has evidently only one coördinate x_0; at such a point x_0, the preceding theorems fail to apply, irrespective of the initial value of y_0. It is evident there are only a finite number of fixed singularities of either the first or second kind.

By the aid of the preceding theorems, and a further theorem first proved by Painlevé,* we can state that a function of x which satisfies the equation $dy/dx = P(x, y)/Q(x, y)$ can be extended over the whole x-plane, and, except at the x-coördinates of the fixed singularities, the only singularities of the function will be poles and branch points. In general such functions will be infinitely many valued.

Let $f(x, x_0, y_0)$ denote that integral of the equation, which at the point x_0

* See Boutroux, *Leçons sur les fonctions définies par les équations différentielles*, p. 15.

has the value y_0. It is shown by various writers, that $f(x, \bar{x}_0, \bar{y}_0)$ which depends obviously upon $\bar{x}_0, \bar{y}_0$ as well as upon x, is algebroid in its three arguments in the neighborhood of x_0, x_0, y_0, provided that x_0, y_0, or x_0, are not the coördinates of a fixed singularity of the first kind, or second kind, respectively.

Furthermore, $f(x, \bar{x}_0, \bar{y}_0)$ is algebroid in its three arguments in the neighborhood of (x_1, x_0, y_0), where x_1 and x_0 are the extremities of a path L which passes through the x-coördinate of no fixed singularity, and along which $f(x, x_0, y_0)$ has no branch point except possibly at x_1. For the purposes of this paper, it is necessary to note further that if at the point x_1, for fixed (x_0, y_0), $f(x, x_0, y_0)$ has a pole or branch point of order k, then there exists a neighborhood of x_1, and one of (x_0, y_0), such that if $(\bar{x}_0, \bar{y}_0)$ is in the latter neighborhood, then $f(x, \bar{x}_0, \bar{y}_0)$ for fixed $(\bar{x}_0, \bar{y}_0)$ will have poles or branch points in the former neighborhood, the sum of whose orders is k. This extension follows immediately from the proof of the preceding statement.

2. Limit solutions and cyclic systems

The integrals of a differential equation, $dy/dx = P(x, y)/Q(x, y)$ are in general infinitely many valued functions over the x-plane. The limiting determinations of the infinitely many determinations of an integral of the differential equation form what we shall call the limit solutions of that integral. Let us give an exact definition:* At a point X, which is not the x-coördinate of a fixed singularity, let $Y_1, Y_2, Y_3, \cdots$ be the determinations of an integral $Y(x)$ counted as distinct if obtained by analytic extension along essentially distinct paths. Then if Z be any limiting value of $Y_1, Y_2, Y_3, \cdots$ the integral $Z(x)$ defined by $f(x, X, Z)$ is a limit solution of $Y(x)$.

Although the definition requires only that infinitely many determinations of $Y(x)$ are near a particular determination, Z, of $Z(x)$ only at X, it readily follows from the algebroid character of $f(x, \bar{x}, \bar{y})$ in its arguments that in the neighborhood of a determination of $Z(x)$ at *any* x not the x-coördinate of a fixed singularity, there are infinitely many determinations of $Y(x)$.

For, the determination Z at the point X can be carried into any other determination at any point x not the x-coördinate of a fixed singularity, over a path L, which avoids all the fixed singularities and branch points of $Z(x)$. Since determinations of $Y(x)$ in any neighborhood of Z exist at X, determinations of $Y(x)$ differing from $Z(x)$ by as little as we please, may be found at any point x on L.

The proof of the following obvious theorem is omitted.

* Cf. Boutroux, Annales de e'école normale supérieure, ser. 3, vol. 27 (1905), p. 448. Comptes Rendus, vol. 138, p. 850–3. *Leçons sur les fonctions définies par les équations différentielles du premier ordre*, pp. 25–30.

THEOREM I. *A limit solution of a limit solution of an integral of the differential equation* $dy/dx = P(x, y)/Q(x, y)$ *is again a limit solution of that integral.*

In general, every integral will have infinitely many limit solutions, each limit solution will have infinitely many limit solutions, etc. In thus forming limit solutions of limit solutions the question naturally arises as to whether there exists a set of integrals such that the limit solutions of any integral of the set are the same set of integrals. An integral with a finite number of values obviously is a special case of such a set. I shall now proceed to show that there exists always a set of integrals having this property.

Let us think of the y-plane as stereographically projected upon the y-sphere in the usual manner. Take any integral $Y(x)$ of the equation

$$dy/dx = P(x, y)/Q(x, y),$$

and consider its various determinations at a point X, not the x-coördinate of a fixed singularity, and suppose these determinations to be marked upon the y-sphere. Suppose likewise that all the determinations at the point X of all the limit solutions of $Y(x)$ are marked upon the y-sphere. The two sets of marked points together form a closed set, of course.

Now, suppose the sphere to be divided up into a finite number of simple sub-regions. A particular limit solution of $Y(x)$ may have determinations lying in every one of the sub-regions of the sphere, or it may fail to have any determination in some of the sub-regions. For our present purposes, it is most convenient to have each sub-region include its boundary, so that a boundary point will lie in two or more regions.

Let $Y_1(x)$ be either $Y(x)$ or a limit solution of $Y(x)$ whose determinations lie in a least number of these sub-regions; that is, any other limit solution of $Y(x)$ must have determinations entering into at least as many sub-regions of the y-sphere as do the determinations of $Y_1(x)$. Having chosen $Y_1(x)$, let us reject from further consideration all those sub-regions of the sphere which contain no determination of $Y_1(x)$. All the determinations of the limit solutions of $Y_1(x)$ will evidently lie only in the sub-regions which have been retained, and will be among the points already marked.

Now, divide up the retained sub-regions into smaller subregions, still following the convention that the boundary of a region is part of that region. Determinations of $Y_1(x)$ or of any particular one of its limit solutions may lie in every one of these smaller subregions, or only in some of them. Let $Y_2(x)$ be one of these integrals, whose determinations enter into a least number of the smaller sub-regions. Now reject from further consideration all the sub-regions which do not contain determinations of $Y_2(x)$.

This process of sub-division and selection of $Y_1(x)$, $Y_2(x)$ $\cdots$ may be

continued indefinitely. If the largest dimension of the largest sub-region in the successive divisions approaches zero, *a minimal set of points, L*, which will be retained through this process, will have the following property.

THEOREM II. *An integral of the equation which at $x = X$ has one of its determinations belonging to a minimal set L must have all its determinations at $x = X$ belonging to the set.*

Proof. First let us remark that a necessary and sufficient condition that a point Z be retained in the process of division and rejection, is that the point Z be a limit point of the determinations of every one of the integrals $Y_1(x)$, $Y_2(x)$, $\cdots$. The sufficiency is obvious, for since the sub-regions are always supposed to retain their boundaries, after the kth division, the limit points of the determinations of $Y_k(x)$ must be retained.

The necessity is shown thus: Suppose that Z were not a limit point of the determinations of $Y_k(x)$. Then we could surround Z by a region T on the sphere, in which there would be no determinations of $Y_k(x)$ and therefore also no determinations of $Y_{k+1}(x)$, $Y_{k+2}(x)$, $\cdots$.

At some division, say the lth, $l > k$, Z would be an interior point of a sub-region lying wholly within T, or would be a boundary point of sub-regions all of which would lie wholly within T. Now since $Y_l(x)$ has no determinations in T, Z would necessarily be rejected with the sub-regions of which it is an interior or boundary point.

Thus we see that if an integral has one of its determinations belonging to L that integral must be a limit solution of $Y_1(x)$, $Y_2(x)$, $\cdots$. But in that case any other determination of the integral is a limit point of determinations of every one of the integrals, $Y_1(x)$, $Y_2(x)$, $\cdots$, and that is sufficient that this determination belong to L.

THEOREM III. *If an integral has determinations at X belonging to a minimal set L, then any limit solution of this integral has at X determinations belonging to the set L.*

For the given integral must be a limit solution of all the integrals $Y_1(x)$, $Y_2(x)$, $\cdots$. Hence so too must be its limit solution. But this suffices to make its limit solution have determinations among the set L.

THEOREM IV. *Any integral which at X has determinations belonging to a minimal set L, is a limit solution of any other integral which at X has determinations belonging to the set L.*

For suppose that $Z_1(x)$ and $Z_2(x)$ are integrals which at X, respectively, have determinations belonging to the set L, and that $Z_1(x)$ is not a limit solution of $Z_2(x)$. Let Z be any determination of $Z_1(x)$. Then at some stage in the process of division and rejection by which the set L was obtained, say the kth stage, Z must lie in a sub-region which contains no determination of $Z_2(x)$ or must be a boundary point of regions which contain

no determinations of $Z_2(x)$. But since $Z_2(x)$ is a limit solution of $Y_k(x)$, its determinations do not enter into more sub-regions of the kth division than do the determinations of $Y_k(x)$. Also since Z is retained throughout the whole process of division and rejection the determinations of $Y_k(x)$ do enter the sub-regions of which Z is an interior or boundary point, but the determinations of $Z_2(x)$ do not. Hence, the determinations of $Z_2(x)$ enter into a fewer number of sub-regions than do those of $Y_k(x)$ and therefore $Y_k(x)$ is not a limit solution of $Y_{k-1}(x)$ which enters into a least number of sub-regions of the kth division. Thus, we get a contradiction.

DEFINITION. *A set of integrals such that any limit solution of any member of the set is again a member of the set, and such that any member of the set is a limit solution of any other member of the set, shall be called a cyclic system.*

Theorems III and IV show the existence of cyclic systems, and it is clear from the method of constructing the set L that *every integral of the equation* $dy/dx = P(x, y)/Q(x, y)$ *has at least one cyclic system of limit solutions.*

The integrals of a cyclic system are essentially alike over any region not including a fixed singularity. For, taking a branch of one integral of a cyclic system in the neighborhood of a point x not the x-coördinate of a fixed singularity, then, from the definition, branches of any other integral of the system may be found which differ indefinitely little from the branch of the chosen integral in the neighborhood of x. Moreover this neighborhood may be taken indefinitely large so long as it includes no fixed singularity.

THEOREM V. *No two different cyclic systems of the same equation can have an integral in common.*

For, from the definition, a cyclic system may be generated from a single one of its members by forming all the limit solutions of that member.

In the next section of this paper, we shall see that, because of this theorem, in a large number of cases there cannot be more than one cyclic system for a single equation. It appears probable that, in general, in these cases, the cyclic system will consist of the totality of integrals of the equation. In these cases every integral is a limit solution of any one integral, so that the determination of a single integral in its entirety is equivalent to determining every integral of the equation.

Examples.

(I) $$\frac{dy}{dx} = y.$$

The general integral is $y = Ce^x$. All the integrals are single valued functions, and therefore each individual integral is a cyclic system by itself. Here, then, we have infinitely many distinct cyclic systems.

(II) $$\frac{dy}{dx} = \frac{\lambda y}{x}.$$

The general integral is

$$y = Cx^{\lambda}.$$

If λ is real and irrational, two distinct integrals, $y = C_1 x^{\lambda}$, $y = C_2 x^{\lambda}$ will be limit solutions of each other if and only if, C_1/C_2 has absolute value equal to unity. In this case if C is any constant, the family of integrals $y = e^{i\phi} Cx^{\lambda}$ obtained by giving ϕ all real values will form a cyclic system. Here, too, we have infinitely many cyclic systems.

(III) $$\frac{dy}{dx} = \frac{y^2}{x}.$$

The general integral is

$$y = -\frac{1}{\log x + C}.$$

The integral $y \equiv 0$ is a limit solution of every integral of the equation. Hence here there is only one cyclic system, and this consists of the integral $y = 0$.

(IV) $$\frac{dy}{dx} = \frac{\lambda_1}{2\pi i}\frac{1}{x - a_1} + \frac{\lambda_2}{2\pi i}\frac{1}{x - a_2} + \frac{\mu_1 i}{2\pi i}\frac{1}{x - a_3} + \frac{\mu_2 i}{2\pi i}\frac{1}{x - a_4}.$$

Here we shall assume that a_1, a_2, a_3, a_4, are distinct, that λ_1 and λ_2 are reals relatively incommensurable to each other, and that μ_1 and μ_2 are reals relatively incommensurable to each other.

The general integral is

$$y = \frac{\lambda_1}{2\pi i}\log(x - a_1) + \frac{\lambda_2}{2\pi i}\log(x - a_2) + \frac{\mu_1 i}{2\pi i}\log(x - a_3) + \frac{\mu_2 i}{2\pi i}\log(x - a_4) + C.$$

If y_0 is a determination of any integral at a point x_0, then all the other determinations of that integral are given by

$$y_0 + (m_1\lambda_1 + m_2\lambda_2) + i(n_1\mu_1 + n_2\mu_2)$$

where m_1, m_2, n_1, n_2, independently take on all integral values positive and negative.

But, inasmuch as λ_1 and λ_2 are relatively incommensurable, we may choose m_1, m_2, so as to make $m_1\lambda_1 + m_2\lambda_2$ indefinitely near to any preassigned real number a. Similarly we may choose n_1, n_2 so as to make $n_1\mu_1 + n_2\mu_2$ indefinitely near to any preassigned real number b.

Hence, at the point x considered, determinations of the integral exist, which come indefinitely near to any previously chosen complex number, Z, whatever. Hence, since the integral considered was any integral, every

integral of the equation has as limit solutions all the integrals of the equation.

Hence, for this equation, the totality of all the integrals forms a cyclic system.

3. Fixed singularities and cyclic systems

The simplest type of fixed singularity, i. e., one where the curves

$$P(x, y) = 0 \quad \text{and} \quad Q(x, y) = 0$$

have a simple intersection, has been much studied. Briot and Bouquet,* in a classic memoir on this subject, proved the following facts. Let $P(x, y)$ and $Q(x, y)$ when expanded into power series about $\bar{x}, \bar{y}$, begin respectively with the terms $a(x - \bar{x}) + b(y - \bar{y}) + \cdots, c(x - \bar{x}) + d(y - \bar{y}) + \cdots$. Then since the intersection is simple, we have $ad - bc \neq 0$. Hence neither root of the equation for λ, $\lambda^2 - (a + d)\lambda + ad - bc = 0$, can be zero. Let the two roots of this equation be λ_1 and λ_2. When these two roots are distinct, the integrals of the equation in the neighborhood of $\bar{x}, \bar{y}$ are given by $g(x, y)^{\lambda_1}/h(x, y)^{\lambda_2} = C$, where C is an arbitrary constant, and $g(x, y)$ and $h(x, y)$ are functions of x, y, and analytic and vanishing at $\bar{x}, \bar{y}$ with non-vanishing jacobian. The functions of x defined by $g(x, y) = 0$ and $h(x, y) = 0$ will be integrals of the differential equation.

Theorem VI.† *If λ_1 and λ_2 are distinct and their ratio is not real, then the two integrals of the differential equation given by $g(x, y) = 0$ and $h(x, y) = 0$ are limit solutions of all the other integrals of the equation which come within a certain neighborhood of $\bar{x}, \bar{y}$.*

Proof. Since the jacobian of $g(x, y)$ and $h(x, y)$ does not vanish at $\bar{x}, \bar{y}$, we may map the neighborhood of $\bar{x}, \bar{y}$ into a neighborhood of $0, 0$, by the transformation $\xi = g(x, y)$, $\eta = h(x, y)$. This transformation will be one to one and analytic. Then the "curves" $g(x, y)^{\lambda_1}/h(x, y)^{\lambda_2} = C$ will go over into the curves $\xi^{\lambda_1}/\eta^{\lambda_2} = C$; or $\eta = C^{-1/\lambda_1}\xi^{\lambda_1/\lambda_2}$. Suppose that the coefficient of the pure imaginary part of the ratio λ_1/λ_2 is positive. If then on any of the "curves" $\eta = C^{-1/\lambda_2}\xi^{\lambda_1/\lambda_2}$ we let ξ circle the origin in the positive sense, staying at a constant distance from it, η will approach the value zero. Hence $\eta \equiv 0$ will be a limit solution of all the curves of the family $\eta = C^{-1/\lambda_2}\xi^{\lambda_1/\lambda_2}$ and therefore the curve $g(x, y) = 0$ will be a limit solution of the family $g(x, y)^{\lambda_1}/h(x, y)^{\lambda_2} = C$. If the coefficient of the pure imaginary in the ratio λ_1/λ_2 is negative, we derive the same conclusion by letting ξ circle the origin in the negative sense. In a similar fashion we

*Journal de l'école polytechnique, vol. 21 (1856), pp. 161–198.

†Cf. Boutroux, *Leçons sur les fonctions définies par les équations différentielles du premier ordre*, p. 116. Rendiconti del Circolo matematico di Palermo, vol. 24 (1907), pp. 209–222.

may show that the curve $h(x, y) = 0$ is a limit solution of every curve of the family.

COROLLARY. *The integrals of only one cyclic system can pass in the neighborhood of a fixed singularity $\bar{x}, \bar{y}$, of the type in Theorem VI, unless the integrals defined by $g(x, y) = 0$, and $h(x, y) = 0$ are finitely many-valued functions of x, in which case each respectively will form a distinct cyclic system.*

For by Theorem VI, all cyclic systems which pass in the neighborhood of $\bar{x}, \bar{y}$, must have as members $g(x, y) = 0$, or $h(x, y) = 0$, and therefore by Theorem V, there cannot be more than two distinct cyclic systems in the neighborhood of $\bar{x}, \bar{y}$. Now suppose $g(x, y) = 0$ yields an infinitely many-valued function, and belongs to a cyclic system. Then there must be infinitely many *other* members of the system which pass arbitrarily near $\bar{x}, \bar{y}$. But by Theorem VI these must have $h(x, y) = 0$ as a limit solution. Hence there cannot be more than one cyclic system of integrals in the neighborhood of $\bar{x}, \bar{y}$. Similarly if $h(x, y) = 0$ is infinitely many valued there cannot be more than one cyclic system in the neighborhood of $\bar{x}, \bar{y}$.

If $g(x, y) = 0$ and $h(x, y) = 0$ are respectively finitely many valued, then each will obviously be a cyclic system.

Example.

$$\frac{dy}{dx} = -\frac{\lambda(x - y) + \mu(x + y)}{\lambda(x - y) - \mu(x + y)}.$$

The conditions of Theorem VI at the point $(0, 0)$ are satisfied if λ/μ is not real.

The general integral is $(y + x)^{\lambda}/(y - x)^{\mu} = C$. In this case

$$g(x, y) = y + x, \quad \text{and} \quad h(x, y) = y - x.$$

Here $g(x, y) = 0$ and $h(x, y) = 0$ are respectively cyclic systems.

4. A FUNDAMENTAL THEOREM

The remainder of this paper is confined to the equation whose fixed singularities are all of the first kind.

If the equation $dy/dx = P(x, y)/Q(x, y)$ is to have no finite singularity of the second kind, it is evidently necessary and sufficient that $Q(x, y)$ shall have no factor of the form $x - x_0$. If there is to be no singularity of the second kind at infinity, it is necessary and sufficient that the degree of $Q(x, y)$ in x, shall exceed that of $P(x, y)$ in x by at least two.

An integral of such an equation which is finitely many valued must be algebraic. For it is everywhere algebroid except possibly at the finite number of fixed singularities; at these however, because they are of the first kind, the integral must remain finite or become infinite. Hence by generalizations of

the theorem on removable singularities, the integral must be algebroid at the fixed singularities, and, being everywhere algebroid, must be algebraic.

We come now to this

FUNDAMENTAL THEOREM, PART I. *If $Y_1(X)$ and $Y_2(X)$ are two integrals of an equation, $dy/dx = P(x, y)/Q(x, y)$, all of whose fixed singularities are of the first kind, and if $Y_1(x)$ has no algebraic limit solution which differs by a constant from an algebraic limit solution of $Y_2(x)$, then either $Y_1(x)$ and $Y_2(x)$ have a limit solution in common, or $Y_1(x)$ and $Y_2(x)$ each pass through a common fixed singularity, i. e., when x tends along some path in the x-plane to the point $\bar{x}$, both $Y_1(x)$ and $Y_2(x)$ will approach the common value $\bar{y}$, where $\bar{x}$, $\bar{y}$, are the coördinates of a fixed singularity.*

PART II. *If $Y_1(x)$ has algebraic limit solutions which differ only by constants from limit solutions of $Y_2(x)$, and if none of these limit solutions are themselves constants, then $Y_1(x)$ and $Y_2(x)$ come arbitrarily near a common fixed singularity.*

Proof of Part I. We shall suppose that $Y_1(x)$ and $Y_2(x)$ have no limit solution in common and show they must then pass through a common fixed singularity.

With this hypothesis we have the following lemmas:

LEMMA I. *Let the x-coördinates of the fixed singularities and the point infinity in the x-plane be surrounded by arbitrary circles, C, respectively. Then there exists a positive number D, and a positive integer k, such that in any circle in the x-plane of radius D, not overlapping the C circles, there are not more than k branch points and zeros of the function $Y_1'(x) - Y_2'(x)$.**

Suppose the lemma were not true. Take a sequence of number pairs, D', k'; D'', k''; D''', k'''; etc., such that the D's are approaching zero, and the k's are becoming infinite. Then for any pair $D^{(i)}$, $k^{(i)}$ it must be possible to find $x^{(i)}$, $y_1^{(i)}$, $y_2^{(i)}$, such that $f_1'(x, x^{(i)}, y_1^{(i)}) - f_2'(x, x^{(i)}, y_2^{(i)})$ has more than $k^{(i)}$ branch points and zeros inside a circle about $x^{(i)}$ of radius $D^{(i)}$, where $f_1(x, x^{(i)}, y^{(i)})$, and $f_2(x, x^{(i)}, y_2^{(i)})$ are respectively branches of $Y_1(x)$ and $Y_2(x)$. Let X be a limit point of the $x^{(i)}$'s. Choose a sequence of $x^{(i)}$'s, which approach X, and let Y_1 be a limiting value of the corresponding $y_1^{(i)}$'s. Choose a sequence of pairs, $x^{(i)}$, $y_1^{(i)}$, such that the x's approach X, and the y's approach Y_1. Let Y_2 be a limit point of the corresponding set $y_2^{(i)}$.

Now $f_1'(x, X, Y_1) - f_2'(x, X, Y_2)$ has at X a branch point of finite order, say the lth, and a zero of finite order, say the mth, provided that

$$f_1'(x, X, Y_1) - f_2'(x, X, Y_2),$$

does not vanish identically. But if $f_1'(x, X, Y_1) - f_2'(x, X, Y_2) \equiv 0$,

* Accents here and later in the paper indicate differentiation.

then integrating, we have $f_1(x, X, Y_1) - f_2(x, X, Y_2) \equiv C$, a constant. This constant cannot be zero since by hypothesis $Y_1(x)$ and $Y_2(x)$ have no limit solution in common. Further, we must have

$$\frac{P(x,f_1)}{Q(x,f_1)} = \frac{P(x,f_2)}{Q(x,f_2)} = \frac{P(x,f_1-c)}{Q(x,f_1-c)} = \frac{P(x,f_2+c)}{Q(x,f_2+c)}.$$

Hence $f_1(x, X, Y_1)$ and $f(x, X, Y_2)$ would be respectively algebraic, and differ from each other only by a constant. But this case is ruled out in the hypothesis of the theorem.

Then by the algebroid character (in the extended sense noted) of $f(x, X, Y)$ in its three arguments, we may find a neighborhood of X, and neighborhoods of X, Y_1, and X, Y_2, such that if $x^{(i)}$, $y_1^{(i)}$, and $x^{(i)}$, $y_2^{(i)}$ be respectively in the latter two neighborhoods, then $f_1'(x, x^{(i)}, y^{(i)}) - f_2'(x, x^{(i)}, y_2^{(i)})$ as a function of x will have branch points and zeros in the former neighborhood, the sum of whose orders does not exceed $l + m$. But this contradicts the defining property of the point X, Y_1, Y_2.

LEMMA 2. *Let the x-coördinates of the fixed singularities, and the point infinity in the x-plane, be surrounded by arbitrary circles C. Then for any positive number E, there exists a positive number M such that if a point x_1 is outside the circles C, and if in a circle of radius E about this point x_1 (this circle not overlapping the C circles), $Y_1'(x) - Y_2'(x)$ has no zero, we have*

$$|Y_1'(x_1) - Y_2'(x_1)| > M.$$

Suppose the lemma were not true. Then, having chosen E arbitrarily, choose a sequence of M's approaching zero: M', M'', etc. Then for any $M^{(i)}$ we can find values, $x^{(i)}$, $y_1^{(i)}$, $y_2^{(i)}$, such that in a circle of radius E about the point x and not overlapping the C circles, there is no zero of the function $f_1'(x, x^{(i)}, y^{(i)}) - f_2'(x, x^{(i)}, y_2^{(i)})$, but nevertheless such that

$$|f_1'(x, x^{(i)}, y_1^{(i)}) - f_2'(x, x^{(i)}, y_2^{(i)})| < M^{(i)}.$$

Let X be a limiting value of the $x^{(i)}$'s. Choose a sequence of $x^{(i)}$'s which approach X. Let Y_1 be a limit value of the corresponding $y^{(i)}$'s. Choose a sequence of pairs, $x^{(i)}$, $y^{(i)}$, such that the x's approach X and the y_1's approach Y_1. Let Y_2 be a limit point of the corresponding $y^{(i)}$'s. Now, just as in proof of the preceding lemma, $f_1'(x, X, Y_1) - f_2'(x, X, Y_2)$ cannot vanish identically. Suppose that $f_1'(x, X, Y_1) - f_2'(x, X, Y_2)$ has a zero at X. Then there is a neighborhood of X, and this neighborhood may be taken arbitrarily small, and a corresponding one of X, Y_1, Y_2, such that if $x^{(i)}$, $y_1^{(i)}$, $y_2^{(i)}$, be in the latter neighborhood then $f_1'(x, x^{(i)}, y_1^{(i)}) - f_2'(x, x^{(i)}, y_2^{(i)})$ has a zero in the former neighborhood. Suppose that

$$f_1'(x, X, Y_1) - f_2'(x, Y, X_2)$$

is not zero at the point X. Then we can find a number m, and a neighborhood of X and a neighborhood of (X, Y_1, Y_2) such that if $x^{(i)}, y_1^{(i)}, y_2^{(i)}$, is in the latter neighborhood, then throughout the former neighborhood $f_1'(x, x^{(i)}, y_1^{(i)}) - f_2'(x, x^{(i)}, y_2^{(i)})$ in absolute value exceeds m. Thus in either case we are led to a contradiction.

We are now ready to prove the first part of the fundamental theorem.

Let $Y_1(x)$ and $Y_2(x)$ be the two integrals having no limit solutions in common. At any point x in the x-plane, not the x-coördinate of a fixed singularity, form the function $w(x) = Y_1(x) - Y_2(x)$ taking any branch of each integral respectively.

Suppose $w(x)$ is laid out on its Riemann surface. The only singularities of $w(x)$ on this surface will be branch points and poles, and singularities in the various sheets at the x-coördinates of the fixed singularities. The totality of these singularities is denumerable.*

Consider the lines of flow of $w(x)$ on the Riemann surface, i. e., those curves along which *arc* w is constant, and consider the analytical extension of such a curve in the direction of decreasing $|w|$. This extension can terminate in these two ways; (1), the curve may approach a point at which $|w|$ approaches zero; (2) the curve may approach a singularity. If a curve exists which terminates in the manner (1), then the main theorem is established. For $w(x) = Y_1(x) - Y_2(x)$ can equal zero only by $Y_1(x)$ and $Y_2(x)$ going through a common fixed singularity. Suppose now there is no curve, *arc* w = *constant*, which terminates in the manner (1). Then at the singularities of $w(x)$, $w \neq 0$, and the values of *arc* w at these singularities is denumerable. But the totality of values of *arc* w on the surface is non-denumerable. Hence we may find curves, *arc* w = *constant* which do not terminate in the manner (2). Since by hypothesis there are no curves which terminate in the manner (1), these curves cannot terminate at all.

Consider such a non-terminating line of flow. Clearly we may surround the point at infinity, and the x-coördinates of the fixed singularities, in the x-plane by circles C, such that this line of flow does not continually remain within one of these circles.

Suppose $|w|$ approaches 0 along the line of flow. Then by reasoning similar to that used in proving the two lemmas, it may be shown that $Y_1(X)$ and $Y_2(X)$ would have a limit solution in common. Hence along the line of flow we have $|w| > d > 0$.

Let ρ be less than the number D of Lemma 1. Clearly ρ may be taken so small, that we may cut an infinite number of distinct segments of the line of flow by circles of radius ρ, having their centers on the line of flow, and not

*Cf. Poincaré, Rendiconti del Circolo matematico di Palermo, vol. 2 (1888), pp. 197–200.

overlapping the C circles. Let such a circle of radius ρ have its center at a point O (see Fig. 1), cutting the segment LM of the line of flow of $|w|$.

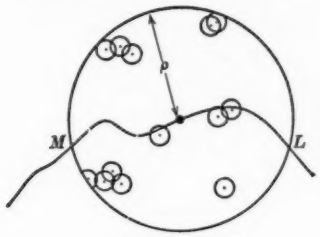

FIG. 1

Then by Lemma I, there are not more than K branch points or poles of $Y_1'(X) - Y_2'(X)$ within this circle.

Choose for the number E of Lemma 2, $\rho/2k$. Surround the branch points and poles of $Y_1(x) - Y_2(x)$ which lie within the circle of radius ρ by circles of radius $\rho/2k$. Then segments of LM will be outside these circles of radius $\rho/2k$, and the combined length of these segments will be not less than $\rho/2$. On these segments by Lemma 2, $|Y_1'(x) - Y_2'(x)| > M > 0$. Hence along these segments,

$$\left|\frac{d \log w}{dx}\right| = \left|\frac{Y_1'(x) - Y_2'(x)}{Y_1(x) - Y_2(x)}\right| > \frac{M}{d}.$$

But along the line of flow,

$$\left|\frac{d \log w}{dx}\right| = -\frac{d \log |w|}{ds} > \frac{M}{d}.$$

Hence along the segment LM, $\log |w|$ diminishes by more than $M\rho/d$.

Hence since an infinite number of segments such as LM may be found along the line of flow of $|w|$, $\log |w|$ decreases indefinitely along the line of flow. But this contradicts the hypothesis that $d > 0$.

Proof of Part II. Suppose that a pair of limit solutions $Z_1(x)$, $Z_2(x)$ of $Y_1(x)$ and $Y_2(x)$ respectively satisfy the relation $Z_1'(x) - Z_2'(x) \equiv 0$ so that Z_1 and Z_2 are algebraic. Then

$$f_1'(x, X, Y_1) - f_2'(x, X, Y_2)$$

may vanish identically. But since $Z_1(x)$ is not a constant it will certainly have a pole for some value of x. But by the relation $Z_1'(x) - Z_2'(x) \equiv 0$ it follows that $Z_2(x)$ also has a pole for this value of x. Hence $Z_1(x)$ and $Z_2(x)$ pass through a common fixed singularity. But now, if a limit solution of an integral passes arbitrarily near to a fixed singularity, then that

integral also must pass arbitrarily near to that fixed singularity. Hence $Y_1(x)$ and $Y_2(x)$ must come respectively arbitrarily near to the common fixed singularity of $Z_1(x)$ and $Z_2(x)$, which proves the second part of the theorem.

If a pair of limit solutions $Z_1(x)$, $Z_2(x)$ reduce to constants, it is no longer true as in the preceding paragraph, that they will pass through a common fixed singularity.

5. The solutions and the fixed singularities. Algebraic solutions

THEOREM VII. *If an integral stay uniformly away from a fixed singularity, then so do all its limit solutions.*

By staying uniformly away, we mean that if $(\bar{x}, \bar{y})$ is the fixed singularity, and $Y(x)$ the integral, then x and y do not come within arbitrarily prescribed neighborhoods of $(\bar{x}, \bar{y})$. With this explanation the theorem becomes obvious.

THEOREM VIII. *If all the fixed singularities of an equation are of the simplest type, and if for all of them the ratio of* λ_1 *to* λ_2 *is not real* (see section 3), *then there cannot be more than one cyclic system of integrals of the equation, unless all the cyclic systems reduce to algebraic integrals.*

For suppose there were two systems, one not reducing to an algebraic integral. Then consider any pair of integrals, taken one from each cyclic system respectively. By Theorems III and V, we see that they cannot have a limit solution in common. They must therefore go through a common fixed singularity. But all the fixed singularities are of the type for which the corollary to Theorem VI applies, and since one of the cyclic systems does not reduce to a finitely valued integral, it must be the only cyclic system in the neighborhood of that fixed singularity.

THEOREM IX. *If all the fixed singularities of an equation are of the simplest type, and if for all of them the ratio of* λ_1 *to* λ_1 *is not real* (see section 3), *then there cannot be more than n algebraic integrals not reducing to constants, where*

$$\frac{n(n-1)}{2} + k = m, \qquad 0 < k \leqq n,$$

where n and k are integers, and m is the total number of fixed singularities.

For these integrals must pass in pairs through different fixed singularities.

We now consider the following important question: Are there any integrals of an equation, which stay uniformly away from all the fixed singularities of the equation? Obviously, an integral y = constant, must always pass through fixed singularities.

THEOREM X. *If there is any integral of the equation which stays uniformly*

away from all the fixed singularities of the equation, then there is a cyclic system all of whose integrals stay uniformly away from the fixed singularities, and there is no other cyclic system of the equation. The integrals of this cyclic system are limit solutions of every integral of the equation.*

For let us form a limiting cyclic system (see Theorem III) from the given integral. This will stay away uniformly from the fixed singularities by Theorem VII. As we have noted, such a cyclic system cannot reduce to a constant. There can be no other cyclic system, for an integral of another system would have to have a fixed singularity in common with every integral of the first cyclic system. Also, since any integral of this system cannot have a fixed singularity in common with any of the integrals of the equation, it must be a limit solution of every integral of the equation.

An algebraic integral is a cyclic system of integrals in itself. Hence we have these corollaries:

COROLLARY I. *If there is an algebraic integral of the equation which passes through a fixed singularity, then there are no integrals of the equation which stay uniformly away from the fixed singularities.*

COROLLARY II. *If there is an integral $y =$ constant, then there are no integrals of the equation which stay uniformly away from the fixed singularities.*

COROLLARY III. *If there are two algebraic integrals of the equation, then there are no integrals of the equation which stay uniformly away from the fixed singularities.*

If the degree of $P(x, y)$ in y does not exceed the degree of $Q(x, y)$ in y by as much as 2, then the transformation, $\bar{y} = 1/y$, will carry the given equation into one which admits the integral, $\bar{y} \equiv 0$. Hence we get the corollary:

COROLLARY IV. *If the degree of $P(x, y)$ in y alone, does not exceed the degree of $Q(x, y)$ in y alone by as much as 2, then there are no integrals of the equation which stay uniformly away from the fixed singularities.*

THEOREM XI. *If the curves $P(x, y) = 0$, and $Q(x, y) = 0$, have only finite intersections in the projective sense, and if all these intersections are simple, then any algebraic integral must pass through a fixed singularity.*

Let the integral be given by the algebraic equation, $F(x, y) = 0$. Let the total degrees of $P(x, y)$, $Q(x, y)$, $F(x, y)$, be respectively, p, q, f. Let the degrees of $P(x, y)$, $Q(x, y)$, $F(x, y)$, $F_x(x, y)$, $F_y(x, y)$ in x alone respectively by $p_1, q_1, f_1, f_{1x}, f_{1y}$, and in y alone respectively by $p_2, q_2, f_2, f_{2x}, f_{2y}$.

The case where $p_2 - q_2 < 2$ has been disposed of by Corollary 4 so that we have the relations

$$p_2 \geqq q_2 + 2, \tag{1}$$

* The restriction is still kept that all the fixed singularities of the equation are of the first kind, see section 5.

$$q_1 \geqq p_1 + 2. \tag{2}$$

Now since $F(x, y)$ represents an integral of the equation

$$PF_y + QF_x \equiv GF, \tag{3}$$

where $G(x, y)$ is a polynomial whose total degree we shall denote by g, and whose degree in x alone and in y alone respectively by g_1 and g_2. We have then these relations,

$$f_{1x} = f_1 - 1, \tag{4}$$

$$f_{2x} \leqq f_2, \tag{5}$$

$$f_{1y} \leqq f_1, \tag{6}$$

$$f_{2y} = f_2 - 1. \tag{7}$$

From (2), (3), (4), (6), we have

$$q_1 + f_1 - 1 \geqq g_1 + f_1 \quad \text{whence} \quad q_1 - 1 \geqq g_1, \tag{8}$$

and from (1), (3), (5), (7),

$$p_2 + f_2 - 1 \geqq g_2 + f_2 \quad \text{whence} \quad p_2 - 1 \geqq g_2. \tag{9}$$

Suppose that $p \geqq q$. Then from (3)

$$p + f - 1 \geqq g + f, \quad \text{or} \quad p - 1 \geqq g. \tag{10}$$

Now if the algebraic integral $F(x, y) = 0$ passes through no fixed singularity, the curve $F(x, y) = 0$, will not pass through any of the intersections of the curves $P(x, y) = 0$, and $Q(x, y) = 0$. Hence from (3), $G(x, y)$ must pass through all the intersections of $P(x, y) = 0$ and $Q(x, y) = 0$, and since these are simple, they will be pq in number. Hence $G(x, y) = 0$ must meet $Q(x, y) = 0$ in pq points, and by (10) we have $pq > gq$. Hence $G(x, y)$ does contain $Q(x, y)$ as a factor, or $G(x, y) \equiv L(x, y)Q(x, y)$. But this contradicts (8).

Similarly we are led to a contradiction if $q > p$.

COROLLARY. *If the curves $P(x, y) = 0$ and $Q(x, y) = 0$ have only simple finite intersections in the projective sense, and if there is a single algebraic integral, then there are no integrals of the equation which stay uniformly away from the fixed singularities.*

ON THE DIFFERENTIAL GEOMETRY OF RULED SURFACES IN 4-SPACE AND CYCLIC SURFACES IN 3-SPACE*

BY

ARTHUR RANUM

INTRODUCTION

This paper is devoted to a further development and application of some of the ideas contained in my two earlier papers *On the projective differential geometry of n-dimensional spreads generated by* ∞^1 *flats*† and *On the projective differential classification of n-dimensional spreads generated by* ∞^1 *flats*,‡ which will be referred to as P. D. G. and P. D. C., respectively.

In P. D. C., §§ 42–59, a study was made of ruled surfaces (2-spreads) in 4-space and their correlative loci, planar 3-spreads. The results there obtained are now supplemented and made more significant by a discussion, in Part I of the present paper, of the relation between these loci and their transversal loci.

The combined and to a certain extent completed theory so developed then finds an application, in Part II, to the differential geometry of cyclic (circled) surfaces and pairs of curves in space of three dimensions. This application depends on the known correspondence between the projective point-geometry of 4-space and the descriptive sphere-geometry of 3-space, by virtue of which a straight line of 4-space represents a circle of 3-space and a plane of 4-space represents a pair of points of 3-space.

Although cyclic surfaces have been much studied before from other points of view, the present method throws a considerable additional light on their structure, as the new theorems will testify. Among the striking results is the fact that there exists on every cyclic surface that is neither annular nor anallagmatic a certain unique and fundamental pair of (quasi-asymptotic) curves. See § 26.

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* Presented to the Society, January 1, 1915.

† Annali di Matematica Pura ed Applicata, ser. 3, vol. 19 (1912), pp. 205–249.

‡ American Journal of Mathematics, vol. 37 (April, 1915).

PART I. TRANSVERSAL RULED SURFACES AND PLANAR 3-SPREADS IN 4-SPACE.

TRANSVERSAL RULED SURFACES

1. Let S be a skew ruled surface immersed in 4-space, and let the homogeneous coördinates of its points be expressed in terms of the parameters u and v by means of the equations

$$X_j = A_j(u) + v\,B_j(u) \qquad (j = 1, \cdots, 5), \tag{1}$$

where $A_j(u)$ and $B_j(u)$ are continuous functions possessing continuous first, second, and third derivatives. Symbolically, we write $S = [A, B]$, and

$$X = A(u) + v\,B(u). \tag{2}$$

The functions A_j and B_j satisfy a linear homogeneous differential equation of the second order, which we assume to be in the canonical form*

$$\frac{d^2 A}{du^2} + \frac{d}{du}(l\,B) + mB = 0$$

or, more briefly,

$$A'' + (l\,B)' + mB = 0, \tag{3}$$

where l and m are functions of u. They also satisfy a differential equation of the third order, but this one of the second order is the only one that we are concerned with at present. The results hold equally well whether we are operating in the real or the complex domain.

2. It is known that in 4-space there exists one and only one line meeting three given skew lines; this we call their *transversal*. Consider three neighboring lines of the surface S, one of which, a, is fixed and non-singular,† and the other two approach the first in any manner along the surface. Their transversal will approach a limiting position c, which we shall speak of as meeting three consecutive lines of S. The locus of c, as a describes S, is a new ruled surface $\bar{S}$, the *transversal* of S.‡ The surface $\bar{S}$ is the only ruled surface whose generators have three-point contact with S.

3. We proceed to set up the actual equations of $\bar{S}$; we shall then discover an interesting relation between S and $\bar{S}$.

Let P be the point of intersection of a and c, and π their common plane; π is obviously tangent to S at P. Let a_1, $a_2 (= a)$, and a_3 be three con-

* See P. D. C., § 42, equation (17′).

† A generator is singular if it either meets its consecutive generator or lies in the 3-flat determined by its two consecutive generators.

‡ Cf. H. Mohrmann, *Ueber die windschiefen Linienflächen im Raume von vier Dimensionen und ihre Hauptangentenflächen als reciproke Linienflächen*, Archiv der Mathematik und Physik, ser. 3, vol. 18 (1911), pp. 66–68; E. Bompiani, *Alcune proprietà proiettivo-differenziali dei sistemi di rette negli iperspazi*, Rendiconti del Circolo Matematico di Palermo, vol. 37 (1914), p. 4.

secutive generators of S, and let a_1, a_2 determine the 3-flat f_{12} and a_2, a_3 the 3-flat f_{23}; a and c will both lie in f_{12}, and also in f_{23}; hence they will lie in the plane of intersection of f_{12} and f_{23}, and the latter must be π. But f_{12} and f_{23} are generators of $S^{1,0}$;* and it follows that π is a generator of the planar developable $S^{1,1}$* and that P is the corresponding point of the curve T.†

Let d be the *characteristic* of the plane π, i. e., the line of intersection of π and its consecutive plane; d is then a generator of $S^{1,2}$. Let b be the tangent to T at the point P, i. e., the corresponding generator of T^1. The lines b and d both pass through P and lie in π. Hence a, b, c, and d belong to a pencil of lines. We wish to prove that *a and c are harmonic conjugates with respect to b and d.*

4. **Proof.** In order to indicate the fact that a is the line joining the points $A_j(u)$ and $B_j(u)$ for a particular value of u, we write

$$a = (A, B). \tag{4}$$

Similarly, $f_{12} = (A, B, A', B')$ and $\pi = (A, B, A')$. In view of (3), it is clear that

$$d = (A, A' + lB). \tag{5}$$

Since P is the point of intersection of d and a, $P = (A)$. Hence

$$b = (A, A'). \tag{6}$$

Since the transversal c belongs to the pencil determined by a and b, it meets the line joining the points (A') and (B) in some point $(A' + kB)$; that is, $c = (A, A' + kB)$, where k is to be determined so that c will have three-point contact with the surface S, and therefore with some curve K on S. The curve K will pass through the point P, for which $v = 0$, and will be given by the equation (2), if we assume u and v in this equation to be functions of a third variable t.

Now three consecutive points of K are determined by (X), $(X_u\,du + X_v\,dv)$, and $(X_{uu}\,du^2 + 2X_{uv}\,dudv + X_{vv}\,dv^2 + X_u\,d^2u + X_v\,d^2v)$, where $X_u = \partial X/\partial u$, etc. By reason of equation (3), these three points become, for $v = 0$, (A), $(A'\,du + Bdv)$, and $(A''\,du^2 + 2B'\,dudv + A'\,d^2u + Bd^2v)$. They are to be collinear and are to lie on the line c. The first two will lie on c, if $dv/du = k$.

In order that the third point may lie on c, it is sufficient to determine $dv/du = k$ so that $A''\,du^2 + 2B'\,dudv = (A'' + 2kB')\,du^2$ will lie in the tangent plane $\pi = (A, B, A')$; for since $A'\,d^2u + Bd^2v$ lies in π, d^2v/d^2u can then be determined so that the third point will be collinear with the

* For the exact definitions of $S^{1,0}$, $S^{1,1}$, T, etc., see P. D. G., § 29, and P. D. C., §§ 42–44.

† Bompiani, loc. cit., pp. 1, 4, 5, calls T a quasi-asymptotic curve, and denotes it by $\gamma_{3,2}$.

other two.* In other words, $A'' + 2kB'$ must be equal to a linear homogeneous function of A, B, and A'. But the only existing relation of this kind is (3); hence $k = \frac{1}{2}l$, and

$$c = (A, A' + \tfrac{1}{2}lB). \tag{7}$$

Comparing (4), (5), (6), and (7), we see that the cross-ratio of the lines a, b, c, d is the same as that of the points A', B, $A' + lB$, and $A' + \frac{1}{2}lB$, which is equal to 2. Hence the lines are harmonic.

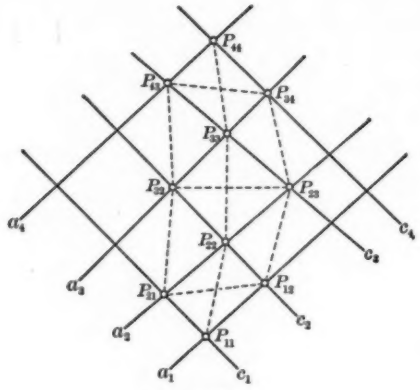

FIG. 1

5. We have now proved the

THEOREM.† *Every non-singular generator of a skew ruled surface S in 4-space and the corresponding generator of its transversal surface $\bar{S}$ are harmonic conjugates with respect to two lines b and d, of which b is the corresponding tangent to the curve of contact of S and $\bar{S}$, and d is the characteristic of their common tangent plane.*

Incidentally, (7) shows that the equations of $\bar{S}$ are

$$X_j = A_j(u) + v[A_j'(u) + \tfrac{1}{2}l(u)B_j(u)] \quad (j = 1, \cdots, 5).$$

6. It may not be amiss to add another proof, or rather quasi-proof, of this theorem, which, although not rigorous, has the advantage of accentuating its geometric significance.

In Fig. 1 let a_1, a_2, a_3, a_4 be four neighboring generators of S; let c_2 be

* Cf. C. L. E. Moore, *Surfaces in hyperspace which have a tangent line with three-point contact passing through each point*, Bulletin of the American Mathematical Society, vol. 18 (1912), p. 284.

† This theorem has also been found by Professor Bompiani. In answer to a recent letter in which I mentioned the theorem to him, and asked him whether he had noticed it, he replied in the affirmative.

the transversal of a_1, a_2, and a_3, meeting them in the points P_{12}, P_{22}, and P_{32}, respectively; and more generally, let c_i be the transversal of a_{i-1}, a_i, and a_{i+1}, meeting them in the points $P_{i-1,\,i}$, P_{ii}, and $P_{i+1,\,i}$. In the figure actual points of intersection are indicated by circles; the others are only apparent. Then c_1, c_2, c_3, c_4 are lines which may be regarded as becoming, by a limit process, generators of $\bar{S}$. Similarly P_{11}, P_{22}, P_{33}, P_{44} may be regarded as becoming points of the curve of contact T; or, if we wish, T may be generated by P_{12}, P_{23}, P_{34}, etc.; $P_{12}\,P_{23}$ then represents one of the tangents to T. Let π_i be the plane determined by a_i and c_i, that is, the common tangent plane to S and $\bar{S}$ at the point P_{ii}; then π_2 is the plane of the quadrilateral $P_{12}\,P_{21}\,P_{32}\,P_{23}$ and π_3 the plane of the quadrilateral $P_{23}\,P_{32}\,P_{43}\,P_{34}$. Hence π_2 and π_3 meet in the line $P_{23}\,P_{32}$, which therefore represents one of the generators of $S^{1,2}$.

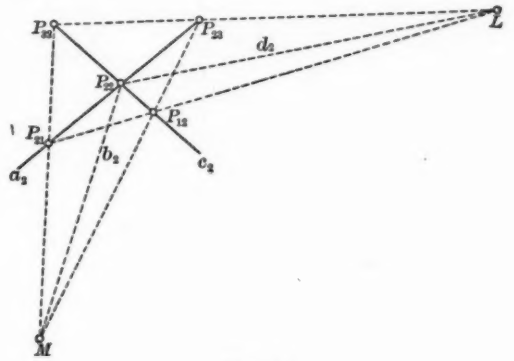

FIG. 2

In Fig. 2 we have the complete quadrangle determined by $P_{12}\,P_{21}\,P_{32}\,P_{23}$. It is evident that the tangent b_2 to T at the point P_{22} can be represented by $P_{22}\,M$ instead of $P_{12}\,P_{23}$, and that the corresponding generator d_2 of $S^{1,2}$ can be represented by $P_{22}\,L$ instead of $P_{23}\,P_{32}$. It follows that a_2, b_2, c_2, d_2 form a harmonic pencil.

7. Let us now examine the significance of this theorem for the three classes (a), (b), (c) of skew ruled surfaces, as they were defined in P. D. C., §§ 45–51.

If S belongs to the most general class (a), T is distinct from the edge of regression $S^{1,3}$ of $S^{1,2}$, and the lines b and d are distinct; hence *the transversal surface* $\bar{S}$ *is another skew ruled surface of the class* (a) and the same subclass (a_{ij}) as S itself; S is then the transversal surface of $\bar{S}$, so that the relation between S and $\bar{S}$ is a reciprocal one. All the skew surfaces therefore whose generators lie in the tangent planes to a developable $S^{1,2}$ and pass through the corresponding points of a curve T lying on $S^{1,2}$ (T not coinciding with

$S^{1,3}$), arrange themselves in pairs of transversals. Every surface of this set touches every other surface of the set along the curve T; while in the case of two transversal surfaces of the set every generator of one has three-point contact with the other.

If S belongs to class (b), T coincides with $S^{1,3}$, and b coincides with d. Hence c also coincides with d, and $\bar{S}$ coincides with $S^{1,2}$; that is, *the transversal surface of a skew ruled surface of class* (b) *is a non-conical developable.* All the surfaces, $\infty^{\prime 1}$ in number, whose generators pass through the points of $S^{1,3}$ and lie in the corresponding tangent planes of $S^{1,2}$, have the same transversal surface $S^{1,2}$. Reciprocally, it is easy to see that all these surfaces, and these alone, are transversal surfaces of $S^{1,2}$. That is, a non-conical developable surface $S^{1,2}$ immersed in F_4 has an infinite number of transversal surfaces, which, apart from $S^{1,2}$ itself, are all skew surfaces of class (b).

If S belongs to class (c), its generators all meet a fixed line d, with which b and c coincide. So the transversal surface $\bar{S} = S^{1,2}$ is of range 0 and reduces to the line d itself.

8. One or two special cases may be mentioned in passing. If T is an orthogonal trajectory of the generators of $S^{1,2}$, the corresponding generators of S and $\bar{S}$ make equal angles with the corresponding tangents to their curve of contact T. If, on the other hand, the corresponding lines of S and $\bar{S}$ are perpendicular to one another, they bisect the interior and exterior angles between the corresponding tangents to T and the generators of $S^{1,2}$.

Moreover, if a and c are constantly parallel, the curve T lies entirely at infinity, the point P and the line b are at infinity, and the line d is parallel to a and c and lies half-way between them. S and $\bar{S}$ then belong to class (a) and subclass (a_{12}) or (a_{22}). Conversely, if $S^{1,2}$ is any developable surface, either non-conical or a point-cone, and if every line a of S is parallel to the corresponding line d of $S^{1,2}$, then every line c of the transversal surface $\bar{S}$ is also parallel to d, and a and c lie on opposite sides of d (in the plane π) and are equidistant from d.

9. In order to illustrate the case in which two transversal surfaces are of class (a), we choose for S the rational quartic surface of subclass (a_{22}) described in P. D. C., § 48, whose equations are

$$x_1 : x_2 : x_3 : x_4 : x_5 = v : 1 : u + v : u^2 + 2uv : u^3 + 3u^2 v. \tag{8}$$

Eliminating u and v, we get the equations

$$x_2 x_4 = x_3^2 - x_1^2, \qquad x_2^2 x_5 = (x_3 - x_1)^2 (x_3 + 2x_1), \tag{9}$$

which represent not only S, but also the plane $x_2 = x_3 - x_1 = 0$, counted twice. The differential equation (3), which the directrices A and B satisfy, is in this case $A'' - B' = 0$. Hence $\bar{S} = [A, A' - \frac{1}{2}B]$, and it follows

that the equations of $\bar{S}$ are obtainable from (8) or (9) by merely replacing x_1 by $-x_1$.

TRANSVERSAL PLANAR 3-SPREADS IN F_4

10. By the use of duality we can now easily develop the corresponding theory of skew planar 3-spreads in F_4 and their transversal 3-spreads. The metrical applications are somewhat more interesting than they were for the ruled surfaces. Let S be any such 3-spread, not a cone. Its simpler properties and classification are given in P. D. C., §§ 52–59. Two consecutive generating planes of S meet in a point, and these points generate a curve $S_{1,0}$.

Now any three planes in F_4 which meet each other by pairs in points determine one and only one plane meeting each of them in a line, namely the plane connecting the three points in which they meet; this plane we call their *transversal*. The transversal of three consecutive planes of S will generate a new planar 3-spread $\bar{S}$, the *transversal* 3-*spread* of S.

Let α be a non-singular plane of S and γ the corresponding plane of $\bar{S}$. They will meet in a line a and determine a 3-flat f; a is then a tangent to the curve $S_{1,0}$, that is, a generator of $S_{1,1}$, while f is the corresponding tangent 3-flat to S (at every point of a), and therefore generates the spread T.* The plane β, in which f meets the consecutive generator of T, is a generator of $T_{1,0}$. Let δ be the corresponding osculating plane of $S_{1,0}$, or generator of $S_{1,2}$. Then β and δ both lie in the 3-flat f and pass through the line a. Hence the four planes $\alpha, \beta, \gamma, \delta$ form a pencil, which by the correlative of the theorem of § 5 we see to be harmonic.

11. Evidently S and $\bar{S}$ touch each other at every point of the developable surface $S_{1,1}$; and f is their common tangent 3-flat at every point of the generator a of $S_{1,1}$; that is, every line lying in f and meeting a has 2-point contact with both 3-spreads. Moreover, every line in the plane γ has 3-point contact with S and every line in the plane α has 3-point contact with $\bar{S}$. Summarizing, we have the

THEOREM. *Every non-singular generator of a non-conical skew planar 3-spread S in 4-space and the corresponding generator of its transversal 3-spread $\bar{S}$ are harmonic conjugates with respect to two planes β and δ, of which δ is the corresponding tangent plane to the developable surface along which S and $\bar{S}$ touch each other and β is the characteristic of their common tangent 3-flat.*

When the equations of S are given in the canonical form† $S = [A, A', B]$, where A and B satisfy a differential equation of the type

$$A''' + (lB)' + mB + nA = 0,$$

then it is easy to see that $\bar{S} = [A, A', A'' + \frac{1}{2}lB]$.

* See P. D. C., § 54.

† See P. D. C., § 52.

12. If S belongs to the class (a), the planes $\alpha, \beta, \gamma, \delta$ are distinct, and $\bar{S}$ is a skew 3-spread belonging to the same class (a) and the same subclass (a_{ij}) as S. Then S is the unique transversal 3-spread of $\bar{S}$.

If S belongs to the class (b), the planes β, γ, δ coincide, and $\bar{S}$ is developable, coinciding with $S_{1,2}$. Conversely, every non-conical developable 3-spread $S_{1,2}$ immersed in F_4 has an infinite number of transversal 3-spreads, which, apart from $S_{1,2}$ itself, are all skew 3-spreads of class (b).

If S belongs to the class (c), its planes all meet a fixed plane δ in lines; β and γ coincide with δ, and $\bar{S}$ degenerates into the plane δ itself.

A fixed 3-flat f', not containing any of the lines of contact a, will intersect S and $\bar{S}$ in two ordinary ruled surfaces R and $\bar{R}$, such that every line of R has 3-point contact with $\bar{R}$ and vice versa; f' will intersect $S_{1,1}$ in the curve of contact of R and $\bar{R}$. The four harmonic planes $\alpha, \beta, \gamma, \delta$ will be met by f' in four harmonic lines a, b, c, d. Since a and c give the two asymptotic directions at a point of R (and of $\bar{R}$), b and d give a pair of conjugate directions.

13. We now proceed to impose certain metrical restrictions on S and $\bar{S}$. If the planes β and δ are perpendicular, they bisect the angles between α and γ; while if α and γ are perpendicular, they bisect the angles between β and δ.

On the other hand if α and γ are constantly parallel, the curve $S_{1,0}$ lies entirely at infinity, the line a and the plane δ lie at infinity, and the plane β is parallel to α and γ, lying halfway between them. Then S and $\bar{S}$ belong to subclass (a_{21}) or (a_{22}). Every plane of S (and of $\bar{S}$) is half-parallel* to its consecutive plane, but not, in general, to its other planes. Conversely, if consecutive planes of S are half-parallel, while non-consecutive planes are not, then every plane of S is completely parallel to the corresponding plane of its transversal 3-spread $\bar{S}$.

Suppose now that any two planes of a planar 3-spread S, whether consecutive or not, are half-parallel. Then the planes of S meet the 3-flat at infinity in a system of lines, every two of which intersect each other; hence they either pass through a fixed point or lie in a fixed plane. In the first case S is a conical (or rather cylindrical) skew planar 3-spread; its planes are half-parallel at the same infinite point. In the second case the planes of S are half-parallel at different infinite points; the curve $S_{1,0}$ generated by the infinite points at which consecutive planes of S are half-parallel is a plane curve. Hence S is a non-conical skew planar 3-spread of class (c).

14. Finally, letting f' (see § 12) be the 3-flat at infinity, we shall consider the case in which R and $\bar{R}$ have certain special properties, while $S_{1,0}$ is a finite curve.

* Half-parallel planes in F_4 meet in an infinite point, while completely parallel planes meet in an infinite line. See Schoute, *Mehrdimensionale Geometrie*, Erster Teil (1902), pp. 20–40.

Now the metrical geometry of f' may be regarded, for our purpose, as a riemannian or elliptic geometry. Hence the interesting case presents itself in which the generators of R are *paratactic** *lines* (clifford parallels), that is, they belong to a right (or left) paratactic congruence. We shall apply the term paratactic also to the planes of S. In other words, *two planes lying in a 4-flat and meeting in a finite point are paratactic, if all the lines of one are equally inclined to the other.*† Obviously, two planes right paratactic to a third plane are right paratactic to each other, whether they have the same points of intersection or not. Hence if *consecutive* planes of S are paratactic, *all* its planes are paratactic. Then S must be a skew 3-spread, but may be a point-cone or a non-conical spread of class (a), (b), or (c).

If it is of class (c), the line d, as defined in § 12, is a fixed line, with which b and c coincide, and the lines a form one regulus (system of generators) R of a clifford surface. If S belongs to the class (a), and to any one of the four subclasses, it may happen that the planes of the transversal 3-spread $\overline{S}$ are left paratactic at the same time that those of S are right paratactic, or vice versa. If so, it is clear that *their loci at infinity, R and $\overline{R}$, are the two reguli of one and the same clifford surface.*

PART II. THE DESCRIPTIVE DIFFERENTIAL GEOMETRY OF CYCLIC SURFACES AND PAIRS OF CURVES IN 3-SPACE. CYCLIC SURFACES

15. By a cyclic (or circled) surface is meant a surface generated by circles. Since a circle may be regarded as the envelope of a pencil of spheres, it is natural to regard the geometry of cyclic surfaces as included in sphere-geometry. The theorems of sphere-geometry are of three essentially different kinds.

First there are the ordinary metrical theorems, which are invariant under the 6-parameter group of movements, which are point-transformations carrying every sphere into a congruent sphere. Then there are the more fundamental properties invariant under the 10-parameter group of conformal transformations, which are also point-transformations carrying spheres into spheres, but not necessarily into congruent spheres. Finally, there are the most fundamental properties of all, the descriptive properties, which are invariant under the 24-parameter group of sphere-transformations; these are not point-transformations at all, but carry point-spheres, in general, into proper spheres.

Our purpose is to study the differential geometry of cyclic surfaces from this last point of view. The previous writers on the subject have confined themselves almost exclusively to metrical or conformal properties. In par-

* See Coolidge, *Non-Euclidean Geometry* (1909), p. 99.

† See Schoute, *Mehrdimensionale Geometrie*, Erster Teil (1902), p. 72.

ticular, Enneper's classification* of cyclic surfaces, as developed by Demartres, Cosserat and Moore,* is invariant under the conformal group, but not under the descriptive group. The purely descriptive classification which we shall find is entirely different from Enneper's except in the broad distinction between annular and non-annular surfaces. We shall find eleven classes of surfaces, of which four are annular and seven non-annular.

16. It will be unnecessary for our purpose to use pentaspherical coördinates. It will be sufficient to write the equation of a sphere A in the form

$$A_1(x^2+y^2+z^2)+A_2x+A_3y+A_4z+A_5=0, \tag{10}$$

and to regard $A_1:A_2:A_3:A_4:A_5$ as its homogeneous coördinates. We shall use the word sphere in a broad sense, to include the plane and the point (point-sphere). The non-singular linear sphere-transformations,

$$kA_i'=\sum_{j=1}^{5}\alpha_{ij}A_j \qquad (i=1,\cdots,5), \tag{11}$$

form the descriptive group under which we are to operate. But the coördinates A_i may be interpreted as representing a point in 4-space F_4, and (11) as a collineation in F_4. Hence the descriptive sphere-geometry of 3-space corresponds to the projective point-geometry of 4-space.

It may be remarked, in passing, that the conformal group in F_3 corresponds to the group of collineations in F_4 that leave a certain quadric 3-spread invariant, and therefore that the conformal geometry of F_3 corresponds to the metrical geometry of a non-euclidean F_4.

Any two spheres A and B determine a pencil of spheres enveloping a circle a, just as two points in F_4 determine a range of points lying on a line. If the coördinates A_i and B_i of these two spheres are functions of a parameter u, the circle a will generate a cyclic surface† S, corresponding to a ruled surface in F_4. Hence the theorems derived in §§ 1–9 of this paper and P. D. C., §§ 42–51, on the projective differential geometry of ruled surfaces in F_4, can be translated immediately into theorems on the descriptive differential geometry of cyclic surfaces in F_3.

17. In the correspondence between these two geometries a point of F_4

*A. Enneper, *Die cyklischen Flächen*, Zeitschrift für Mathematik und Physik, vol. 14 (1869), pp. 393–421. G. Demartres, *Sur les surfaces à génératrice circulaire*, Annales de l'école normale supérieure, ser. 3, vol. 2 (1885), p. 142. E. Cosserat, *Sur le cercle considéré comme élément générateur de l'espace*, Annales de Toulouse, vol. 3 (1889), E, pp. 1–12, 25–27. C. L. E. Moore, *Lines in space of four dimensions and their interpretation in the geometry of the circle in space of three dimensions*, American Journal of Mathematics, vol. 33 (1911), pp. 143–147. See also Demartres, *Cours de géométrie infinitésimale* (1913), pp. 308–311, and R. V. Lilienthal, *Vorlesungen über Differentialgeometrie*, zweiter Band, erster Teil (1913), pp. 102–115.

† Naturally circles here include straight lines, and cyclic surfaces include ruled surfaces.

corresponds to a sphere of F_3, a line of F_4 to a circle (pencil of spheres) of F_3, a plane of F_4 to a pair of points of F_3; the points of a plane correspond to the spheres of a linear congruence, or net, which are the spheres orthogonal to a given circle, or passing through a given pair of points, the foci of the given circle; the lines of the plane correspond to the circles passing through the given pair of points, or in bi-involution* with the given circle. A pencil of lines corresponds to a pencil of circles (lying on a sphere and passing through two points of the sphere). The points of a 3-flat correspond to the spheres of a linear complex, which are the spheres orthogonal to a given sphere; the planes of the 3-flat correspond to the pairs of points inverse with respect to the given sphere. A pencil of planes (passing through a line and lying in a 3-flat) corresponds to a *pencil of point-pairs*, by which is meant a system of point-pairs on a circle, which are inverse with respect to a given sphere orthogonal to the circle; the lines joining the pairs of points therefore pass through the center of the given sphere.

Orthogonality, to be sure, is not a descriptive property of spheres and circles, and so, properly speaking, has no place in this discussion. Nevertheless, we have introduced it for convenience in the statement of certain results. It will be readily seen that orthogonality is not essential to the argument that is to follow, and that by circumlocution all mention of it could be avoided.

On the basis of the known classification of ruled surfaces in F_4, we proceed to find the corresponding classification of cyclic surfaces in F_3. Let Σ and S be corresponding ruled and cyclic surfaces, respectively. The surface Σ will be a developable or a skew surface, according as every two consecutive lines meet in a point or not. Hence S will be an annular or a non-annular surface, according as every two consecutive circles are cospherical or not.

Annular surfaces

18. Since a developable Σ is either a point-cone or is generated by the tangents to a curve Σ_1, therefore an annular surface S is either a sphere or the envelope of a family S_1 of ∞^1 spheres.

In the latter case Σ_1 is either a plane curve, a 3-space curve or a 4-space curve; that is, Σ belongs to one of the types† $(01\bar{1}0)$, $(01\bar{1}10)$, or $(01\bar{1}\bar{1}10)$. In F_3 this means that any two consecutive circles, a and b, of S determine a net of spheres containing the pencils of spheres passing through a and b, and that if this net is fixed, S is of type $(01\bar{1}0)$. If this net is variable, then three consecutive circles of S will similarly determine a linear complex of spheres, which, if fixed, gives the type $(01\bar{1}10)$, and if variable, the type $(01\bar{1}\bar{1}10)$.

* See Koenigs, *Contributions à la théorie du cercle dans l'espace*, Annales de Toulouse, vol. 2 (1888), F, p. 9.

† See P. D. C., §§ 33–35.

Accordingly, annular surfaces enveloped by ∞^1 spheres are of three distinct types, as follows.

19. **Type** $(01\bar{1}0)$. Since Σ^1 is a fixed plane, S^1 is a fixed pair of points, and S may be described as a surface generated by circles passing through two fixed points, or in bi-involution with a fixed circle c. If one of the fixed points is at infinity, the generators of S become straight lines and S becomes a cone. If the fixed circle c is a straight line, S becomes a surface of revolution; whereas, if c is not a straight line, S may be described as an *anallagmatic surface whose deferent is a plane curve.* If the deferent is a conic, the surface becomes a cyclide of Dupin.

20. **Type** $(01\bar{1}10)$. Here S^1 consists of a pair of curves enveloped by the circles of S. Hence S is generated by a circle constantly tangent to a pair of curves. One of these curves may degenerate into a fixed point. Moreover, since Σ^2 is a fixed 3-flat, the circles of S are all orthogonal to a fixed sphere σ, and the point-pairs of S^1 are inverse with respect to σ. If σ is not a plane, S may be described as an *anallagmatic surface whose deferent is a space curve.* The simplest algebraic example is the one in which the deferent is a twisted cubic. If one of the component curves of S^1 degenerates into the point at infinity, S becomes a developable surface.

21. **Type** $(01\bar{1}110)$. The circles of S are again tangent to a pair of curves S^1, as in the preceding case, but are now no longer orthogonal to a sphere; so S cannot be anallagmatic. Neither component curve of S^1 can degenerate into a point; S cannot be a ruled surface. Every three consecutive circles of S determine one sphere to which they are orthogonal, and these spheres envelope another annular surface T, also of type $(01\bar{1}110)$, correlative to S. Every circle of T is in bi-involution with two consecutive circles of S, and conversely. The foci of every circle of T are therefore the corresponding pair of points of S^1.

It may happen that every point-pair of S^1 is a pair of coincident points, and that the generators of S are the osculating circles of a single curve S^1. But this special case, while listed as a distinct type in Enneper's classification, is not a distinct type in our descriptive classification; for a pair of coincident points will not be carried into a pair of coincident points by the general sphere-transformation.

Non-annular cyclic surfaces

22. In this case two consecutive circles of S are not, in general, cospherical, and therefore cannot meet in a pair of points. They may meet in a single point, to be sure, but that is not a descriptive property, and so does not determine a distinct class of surfaces from our standpoint.

In F_4 the corresponding ruled surface Σ is skew, and so has ∞^2 tangent

planes, as well as ∞^2 points. Every plane containing a given generator is tangent to the surface at some point of the generator; and these planes are projectively related to their points of contact. Hence S is enveloped in a very special manner* by the system of ∞^2 spheres passing through its generating circles; this system therefore consists of ∞^1 pencils. Every sphere of the system is a bitangent to the surface, touching it at a pair of points on the generator through which it passes. The pairs of points of contact of the spheres of a pencil containing a given circle of S form a pencil of point-pairs on the circle, projective to the pencil of spheres.

Just as two consecutive lines of Σ determine a 3-flat f, so two consecutive circles of S determine a linear complex of spheres containing the pencils of spheres passing through them, and therefore also a single sphere σ to which they are orthogonal. The point-pairs of the pencil just mentioned are inverse with respect to σ. If f is fixed, Σ is a 3-space surface, of type $(0\bar{2}0;)$† while if f is variable, Σ is a 4-space surface, of type $(0\bar{2}10)$.† This gives us the two principal classes of non-annular cyclic surfaces S. They are of type $(0\bar{2}0)$ or of type $(0\bar{2}10)$, according as their generators are orthogonal to a fixed sphere or not.

23. **Type** $(0\bar{2}0)$. The surface S may be described, in general, as an *anallagmatic surface, whose deferent is a skew ruled surface.* An exceptional case is that in which S has a plane of symmetry, to which its generators are orthogonal. Type $(0\bar{2}0)$ includes the most general class of cyclic, anallagmatic surfaces. The simplest algebraic example is that in which the deferent is a quadric surface and S is a cyclide; Σ is then also a quadric surface. If the fixed orthogonal sphere σ is a point-sphere, S is generated by circles passing through a fixed point, and in particular, if σ is at infinity, S degenerates into a skew ruled surface.

Since three consecutive lines of Σ determine an osculating quadric surface $\bar{\Sigma}$, and have an infinite number of common transversals, which are the generators of the other system of $\bar{\Sigma}$, therefore three consecutive circles a, b, c, of S determine an osculating cyclide $\bar{S}$ and have an infinite number of *transversal circles,* namely circles cospherical with a, with b, and with c; these transversals are the generators of the other system of the cyclide $\bar{S}$.

Type $(0\bar{2}10)$

24. We now come to the most general type of cyclic surfaces S, those which are neither annular nor anallagmatic. Two consecutive circles of S are not, in general, cospherical, and three consecutive circles are not, in general,

* See Demartres, loc. cit., pp. 147–148.

† See P. D. C., §§ 36 and 39.

orthogonal to the same sphere. Let

$$A_1(x^2+y^2+z^2)+A_2x+A_3y+A_4z+A_5=0 \tag{10}$$

and

$$B_1(x^2+y^2+z^2)+B_2x+B_3y+B_4z+B_5=0 \tag{12}$$

be the equations of any two spheres whose circle of intersection is a generator of the surface S. Therefore A_j and B_j $(j=1,\cdots,5)$ are to be regarded as functions of a parameter u. Symbolically, we write $S=[A,B]$. The functions A_j, B_j are five linearly independent solutions of a certain differential equation of the second order, which can, by a proper choice* of the spheres A and B, be written in the canonical form

$$\frac{d^2A}{du^2}+\frac{d}{du}(lB)+mB=0. \tag{13}$$

We remark in passing that if S were a surface of type $(0\bar{2}0)$, A_j and B_j would satisfy *two* independent differential equations of the second order, while if S were an annular surface, A_j and B_j would satisfy a differential equation of the *first* order.

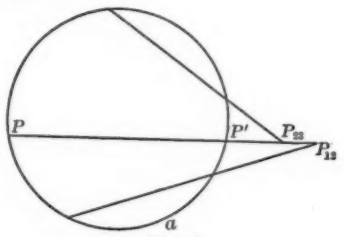

FIG. 3

25. If a_1, $a_2\,(=a)$, and a_3 are three consecutive non-singular circles of S, let P_{12} be the center of the sphere σ_{12} that is orthogonal to a_1 and a_2, and let P_{23} be the center of the sphere σ_{23}, orthogonal to a_2 and a_3. Then the lines lying in the plane of a and passing through P_{12} will meet a in a pencil of point-pairs, inverse with respect to σ_{12}; similarly the lines through P_{23} will meet a in another pencil of point-pairs inverse with respect to σ_{23}. These two pencils will have one point-pair in common, namely the pair of points P, P' (see Fig. 3), in which the line $P_{12}P_{23}$ meets a. The net of spheres passing through P and P' is common to the two linear complexes determined by a_1, a_2 and a_2, a_3, respectively. The points P, P' will generate a pair of curves $S^{1,1}$ lying on S, and represented, in F_4, by the planar developable 3-spread $\Sigma^{1,1}$. Hence we can write

$$S^{1,1}=[A,A',B], \tag{14}$$

* See § 1, equation (3).

thus indicating that P, P' are the points of intersection of the three spheres A, A', and B. The sphere passing through a and touching the surface S at the points P, P' is obviously A itself.

26. Now the component curves of $S^{1,1}$ are so related that they have a common tangent circle d at the points P, P'; this circle (see Fig. 4) lies on the sphere A, and either generates an annular surface

$$S^{1,2} = [A, A' + lB], \tag{15}$$

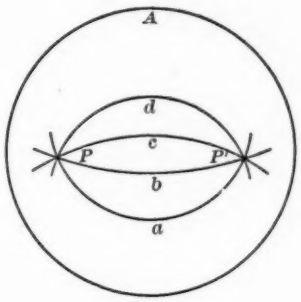

FIG. 4

or is fixed and coincides with $S^{1,1}$. Furthermore, the spheres A either envelope an annular surface $T = [A, A']$, or form a pencil; that is, the characteristic circle b of the sphere A either generates the annular surface T or is fixed and coincides with d; in either case b evidently passes through P and P'.

By translating the theorems of P. D. C., §§ 43 and 44, into the language of circle-geometry, we see that *on every cyclic surface S that is neither annular nor anallagmatic there exists a unique pair of curves* $S^{1,1}$, *such that at every pair of points in which they meet the same generator of S they have a common tangent circle,* and also that *there exists a unique system of* ∞^1 *spheres A bitangent to S such that their points of contact generate* $S^{1,1}$.

We shall call $S^{1,1}$ the pair of *quasi-asymptotic curves* on S. It is clear that neither component of $S^{1,1}$ can degenerate into a fixed point. The variable points P and P' may, however, coincide, so that $S^{1,1}$ becomes a *double* quasi-asymptotic curve; the four circles a, b, c, d are then mutually tangent at the same point. In particular, this will always happen, when every two consecutive circles of S meet in a point P; for then σ_{12} and σ_{23} (§ 25) are point-spheres lying on a. The double quasi-asymptotic curve on S is in that case the locus of the point P.

27. Finally, it is clear that a non-singular circle a and two consecutive circles of S have one and only one *transversal circle* c (cospherical with each of the three), instead of an infinite number, as in the case of an annular or

anallagmatic surface; c will generate a new cyclic surface $\bar{S}$, the transversal of S. From the developments of §§ 2–6 it follows that c lies on the sphere A and passes through the points P, P', that the four circles a, b, c, d form a harmonic pencil (see Fig. 4), and that

$$\bar{S} = [A, A' + \tfrac{1}{2}lB]. \tag{16}$$

The transversal c has 3-point contact with S at each of the points P, P', while b and d have ordinary two-point contact with S at these points.

Hence *two transversal surfaces S and $\bar{S}$ are so related that every circle of one has 3-point contact with the other at each of a pair of points (which may coincide), and that these point-pairs generate the quasi-asymptotic curves $S^{1,1}$ on S.* The harmonic property can be expressed by the

THEOREM. *Every non-singular circle of a cyclic surface S that is neither annular nor anallagmatic and the corresponding circle of its transversal surface $\bar{S}$ are harmonic conjugates with respect to two circles b and d, of which d is the common tangent circle to the two curves of contact of S and $\bar{S}$ at their corresponding points of contact, and b is the characteristic circle of the common bitangent sphere to S and $\bar{S}$ at these points.*

The three principal classes of cyclic surfaces of type $(0\bar{2}10)$ depend on the values of l and m in equation (13), as follows:

Class (a), $l \neq 0$. Class (b), $l = 0$, $m \neq 0$. Class (c), $l = m = 0$.

28. **Class** (a). In this case the four circles a, b, c, d (Fig. 4) are all distinct, and the cyclic surfaces S, T, $\bar{S}$, $S^{1,2}$, generated by them, are therefore distinct. Two consecutive circles d and d' of the annular surface $S^{1,2}$ determine a sphere*

$$C = A' + lB - \frac{m}{l}A, \tag{17}$$

which may be fixed or variable, but which cannot coincide with A, and cannot be tangent to S. Two consecutive circles b and b' of the annular surface T intersect in a pair of points which may or may not be inverse with respect to a fixed sphere, but cannot coincide with P, P', and cannot lie on S. These points generate a pair of curves

$$T^1 = [A, A', (lB)' + mB], \tag{18}$$

associated with S, but not lying on it.

A cyclic surface S of class (a) is evidently characterized by the fact that *every common tangent circle d to its quasi-asymptotic curves at a pair of corresponding points is distinct, in general, from the characteristic circle b of the sphere bitangent to S at those points.*

* See P. D. C., § 45.

Among the circles of the pencil lying on the sphere A and passing through the points P, P' just two generate annular surfaces, namely b and d. The rest generate surfaces of type $(0\bar{2}10)$ and class (a), like S. In particular, *the transversal surface* $\bar{S}$ *is of class* (a), and *its* transversal surface is S itself. All the non-annular surfaces generated by circles lying on the spheres A and passing through the point-pairs P, P' arrange themselves in pairs of transversals.

29. Subclass (a_{11}). This is the most general kind of cyclic surfaces. In this case $S^{1,2}$ is enveloped by ∞^1 spheres C, and is therefore of type $(01\bar{1}110)$; T is also of type $(01\bar{1}110)$; its generators are not orthogonal to a fixed sphere.

Subclass (a_{12}). Again $S^{1,2}$ is of type $(01\bar{1}110)$, while T is now anallagmatic, of type $(01\bar{1}10)$; its generators b and the spheres A are all orthogonal to a fixed sphere σ; the pair of curves T^1 are inverse with respect to σ.

Subclass (a_{21}). Again T is of type $(01\bar{1}110)$, while C (see (17)) is now a fixed sphere, with which $S^{1,2}$ coincides. The circles d and the point-pairs P, P' all lie on the sphere C; so that the quasi-asymptotic curves $S^{1,1}$ are spherical curves.

Subclass (a_{22}). Here $S^{1,2}$ is a sphere, and T is an anallagmatic surface of type $(01\bar{1}10)$.

If the spheres A are planes, S is necessarily of subclass (a_{12}) or (a_{22}); for then T is a developable surface. Its edge of regression (the finite component of T^1) is the locus of the center Q of the sphere which is orthogonal to a and its consecutive circle of S. That is, the pairs of points of contact with S of the spheres passing through a are concurrent with Q.

30. **Class** (b). The circles b and c coincide with d, and the cyclic surfaces T and $\bar{S}$ therefore coincide with $S^{1,2}$. The latter is an annular surface of type $(01\bar{1}110)$, and is therefore neither a sphere nor an anallagmatic surface. The spheres C coincide with the spheres A, and the pair of curves T^1 coincide with the pair of quasi-asymptotic curves $S^{1,1}$.

A cyclic surface S of class (b) is characterized by the fact that *its quasi-asymptotic curves* $S^{1,1}$ *are non-spherical and that every common tangent circle to* $S^{1,1}$ *at a pair of corresponding points is also the characteristic of the sphere bitangent to* S *at those points.*

It is not difficult to show that the quasi-asymptotic curves are in this case ordinary asymptotic curves, while for surfaces of class (a) they are not asymptotic curves.

Since $\bar{S}$ coincides with $S^{1,2}$, *the transversal surface of a cyclic surface* S *of type* $(0\bar{2}10)$ *and class* (b) *is an annular surface of type* $(01\bar{1}110)$, and its generators are the common tangent circles to the quasi-asymptotic curves on S. Conversely, every annular surface $\bar{S}$ of type $(01\bar{1}110)$ has an infinite number of transversal surfaces of class (b). The generators of every one of these lie

on the spheres enveloping $\bar{S}$ and pass through the corresponding point-pairs of the curves enveloped by the generators of $\bar{S}$.

31. **Class** (c). The circles b and c again coincide with d; but the latter is now a fixed circle. Hence the quasi-asymptotic curves both coincide with d, and the spheres A form a pencil passing through d.

A cyclic surface of class (c) is characterized by the fact that *its generators are cospherical with a fixed circle d without lying on a fixed sphere and without being orthogonal to a fixed sphere.* Its transversal surface degenerates into the single circle d.

32. **Illustration of class** (a). When a cyclic surface S of class (a) is real, it may happen that one or more of its associated pairs of curves $S^{1,1}$, T^1, or its associated surfaces $S^{1,2}$, T, $\bar{S}$, are imaginary. The fact that all of them may, however, be real will be made clear by the following example.

Consider a helicoidal movement, one of whose path-curves is the helix H whose equations are $x = r\cos\theta$, $y = r\sin\theta$, $z = k\theta$, where

$$k = r\tan\alpha. \tag{19}$$

Also consider a circle a of radius r, whose center is at the origin, whose plane makes an angle $\pi/4$ with the xy-plane, and which has one diameter on the x-axis; a will meet the helix at the point $(r, 0, 0)$. Let S be the surface generated by the circle a as it is carried along by the helicoidal movement.

33. Then it is clear that as θ varies, the plane

$$A : x\sin\theta - y\cos\theta + z - k\theta = 0$$

and the sphere

$$B : x^2 + y^2 + (z - k\theta)^2 = r^2$$

will constantly meet in a generating circle a of S. Writing the homogeneous coördinates (see § 16) of A and B, we have

$$A = (0, \sin\theta, -\cos\theta, 1, -k\theta),$$

$$B = (1, 0, 0, -2k\theta, k^2\theta^2 - r^2).$$

Differentiating with respect to θ, we find that the pairs of corresponding coördinates of A and B satisfy the differential equation

$$A'' + A + \frac{1}{2k}B' = 0.$$

Although this equation is not quite in the canonical form (13), it is easy to see that the presence of the extra term A will not affect the applicability to this example of the results reached above. Hence $l = 1/2k$, $m = 0$, and S belongs to class (a). In order to find $S^{1,2}$ and $\bar{S}$, we shall need, in view of (15) and (16), to find the spheres $A_1 = A' + B/2k$ and $A_2 = A' + B/4k$.

34. Let us now fix upon a particular generator, $\theta = 0$, of S, and write the corresponding equations of the seven spheres A, B, A', B', A'', A_1, and A_2, referred to rectangular coördinates, as follows:

$$A : y = z, \qquad B : x^2 + y^2 + z^2 = r^2,$$

$$A' : x = k, \qquad B' : z = 0, \qquad A'' : y = 0,$$

$$A_1 : (x + k)^2 + y^2 + z^2 = r^2 + 3k^2,$$

$$A_2 : (x + 2k)^2 + y^2 + z^2 = r^2 + 8k^2.$$

The four spheres B, A', A_2, A_1 intersect the sphere (plane) A in four circles a, b, c, d, respectively, of which b is a straight line. These circles all pass through the pair of points P, P', whose coördinates are

$$[k, \pm\sqrt{\tfrac{1}{2}(r^2 - k^2)}, \pm\sqrt{\tfrac{1}{2}(r^2 - k^2)}].$$

Their centers are on the x-axis and are situated at the distances 0, ∞, $-2k$, $-k$, respectively, from the origin; this verifies the harmonic property. These circles, and therefore the surfaces S, T, $\bar{S}$, $S^{1,2}$, which they generate, are necessarily real. The surface T is a developable, whose edge of regression $T^1 = [A, A', A'']$ is a helix generated by the point $(k, 0, 0)$.

The generator d of $S^{1,2}$ is constantly tangent to the pair of quasi-asymptotic curves $S^{1,1} = [A, A', B]$, generated by P and P'; these are also helices, and are real and distinct, if $k < r$, that is, in view of (19), if the angle α of the helix H is $< \pi/4$. They coincide with each other and with H, if $\alpha = \pi/4$, and become conjugate imaginary curves, if $\alpha > \pi/4$. If $\alpha \neq \pi/4$, the circle c has 3-point contact with S at P and also at P'. Since $S^{1,2}$ is not a sphere, and since T is of type $(01\bar{1}10)$, S and $\bar{S}$ belong to the subclass (a_{12}).

35. **Illustration of class** (b). Out of the material furnished by the preceding example, we can easily construct an example of a cyclic surface of class (b). For since $S^{1,2}$ is an annular surface of type $(01\bar{1}110)$, it is clear that any circle lying on the sphere A_1 and passing through the points P, P', except d, will sweep out, under the influence of the helicoidal movement, a surface of the kind required. In particular, the circle in which the plane $x = k$ meets A_1 will give a simple example.

Pairs of curves in sphere-geometry

36. In the correspondence which we have set up between the projective point-geometry of F_4 and the descriptive sphere-geometry of F_3 a plane of F_4 represents a pair of points of F_3; hence a planar 3-spread represents a pair of curves between which there exists a continuous one-to-one point-correspondence. The two curves may be two parts of the same curve, or, again,

they may be coincident curves, by which is meant that they are generated by pairs of coincident points. On the basis of the study of planar 3-spreads given in P. D. C., §§ 52–59 and in §§ 10–14 of this paper, we shall briefly sketch the descriptive differential sphere-geometry of pairs of curves.

Just as ruled surfaces and planar 3-spreads are dual loci in F_4, so cyclic surfaces and pairs of curves are dual loci in the sphere-geometry of F_3. This duality may be made precise by considering a circle and its foci; as the circle generates a surface, its foci describe a pair of curves,* and vice versa.

Let C^1 and C^2 be any two curves of F_3; and let C denote the two curves taken together, when a continuous point-correspondence has been set up between them. Let P^1 and P^2 be any non-singular pair of corresponding points of C^1 and C^2. We shall call C an annular pair of curves, when C^1 and C^2 have a common tangent circle at every pair of corresponding points P^1, P^2. We proceed to the classification of annular pairs of curves, and then take up the non-annular kind.

37. **Annular pairs of curves.** If the common tangent circle a to C^1 and C^2 is fixed, C^1 and C^2 will simply coincide with parts of the circle a. Apart from this trivial case, C is the envelope of a family of circles, and belongs to one of the four following types.

TYPE $(01\bar{1}0)$. Here C is a pair of spherical and anallagmatic curves. (We shall call two curves *anallagmatic*, if their corresponding points are inverse with respect to a fixed sphere.) Hence C is a pair of curves lying on a fixed sphere and inverse with respect to a fixed circle of the sphere.

TYPE $(01\bar{1}10)$. In this case C is spherical and annular, but not anallagmatic. Hence the enveloping circles cannot be straight lines.

TYPE $(011\bar{1}0)$. Here C is anallagmatic and annular, but not spherical. This is the most general type of anallagmatic curves, and includes the special case in which C has a plane of symmetry, and also the special case in which one of the component curves of C degenerates into a fixed point.

TYPE $(011\bar{1}10)$. Here C is annular, but neither spherical nor anallagmatic. This is the most general type of annular pairs of curves.

NON-ANNULAR PAIRS OF CURVES

38. When C is non-annular, C^1 and C^2 have just one common tangent sphere σ at every pair of corresponding points; σ may be fixed or variable. Thus we have two types.

TYPE $(0\bar{2}0)$. Here S is a pair of curves lying on a fixed sphere, but is neither annular, nor anallagmatic. This is the most general type of spherical curves.

* Cf. Laguerre, *Mémoir sur l'emploi des imaginaires dans la géométrie de l'espace*, Nouvelles annales de mathématiques, ser. 2, vol. 11 (1872), pp. 14–18.

39. Type $(01\bar{2}0)$. In this case C is neither annular nor spherical. Two consecutive point-pairs of C are not concyclic and three consecutive point-pairs are not cospherical. Consider three consecutive point-pairs of C, of which the intermediate one is $P = (P^1, P^2)$, and let σ_{12} be the sphere determined by the first and second of these, and σ_{23} the sphere determined by the second and third. Also let a be the circle of intersection of σ_{12} and σ_{23}; a passes through P^1 and P^2, and is cospherical with its consecutive circle; hence it generates an annular surface $C_{1,1}$. *Associated with every pair of curves C of type $(01\bar{2}0)$ is a unique annular surface $C_{1,1}$ whose generating circles pass through the point-pairs of C.*

The sphere determined by the first and third of the above consecutive point-pairs will meet a in a point-pair $R = (R^1, R^2)$, which is the *transversal* of the three point-pairs, in the sense that it is concyclic with each of them; R will describe the transversal pair of curves $\overline{C}$.

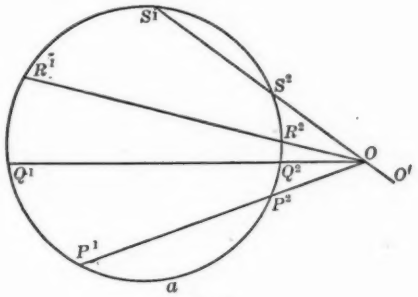

Fig. 5

Let $Q = (Q^1, Q^2)$ be the point-pair in which a meets its consecutive generator of $C_{1,1}$. The three point-pairs P, Q, R (see Fig. 5) belong to a pencil of point-pairs whose vertex O is the center of a sphere with respect to which they are inverse. Its consecutive pencil of point-pairs, whose vertex is O', will have in common with the original pencil just one point-pair $S = (S^1, S^2)$, which is collinear with both O and O'. The four point-pairs P, Q, R, and S form a *harmonic pencil* on the circle a, in the sense that the lines joining them are harmonic.

40. **Class** (a). This is the general case in which the point-pairs P, Q, R, and S are all distinct and describe four pairs of curves C, $C_{1,2}$, $\overline{C}$, and D, respectively, of which two are annular, $C_{1,2}$ and D. The annular pair $C_{1,2}$ described by (Q^1, Q^2) is either of type $(011\bar{1}10)$ or anallagmatic, of type $(011\bar{1}0)$. On the other hand, the annular pair D described by (S^1, S^2) is either of type $(011\bar{1}10)$ or spherical, of type $(01\bar{1}10)$. The tangent circle to D at the points S^1, S^2 lies on the sphere that envelopes $C_{1,1}$ along the circle a.

The transversal pair of curves $\bar{C}$ is of class (a). Every point-pair lying on a and belonging to the pencil determined by Q and S (except Q and S themselves) will describe a pair of curves of class (a).

41. **Class** (b). The point-pairs R and S coincide with Q, and the annular pair of curves $C_{1,2}$ described by Q is of the general type $(011\bar{1}10)$. Hence $C = S_{1,2}$, that is, the transversal pair of curves is annular. Every point-pair lying on a and belonging to the pencil determined by Q and its consecutive point-pair (except Q itself) will describe a pair of curves of class (b), and having $S_{1,2}$ for its transversal pair of curves.

42. **Class** (c). The point-pairs R and S coincide with Q, and Q is fixed. Hence C may be described as a non-spherical, non-anallagmatic pair of curves, whose point-pairs are all concyclic with a fixed point-pair. The transversal pair of curves degenerates into this fixed point-pair.

CORNELL UNIVERSITY,
ITHACA, N. Y.

QUARTIC CURVES MODULO 2*

BY

L. E. DICKSON

1. **Introduction.** Let $f(x, y, z)$ be a homogeneous form of order n with integral coefficients. The points for which the three partial derivatives of f are congruent to zero modulo 2 shall be called derived points. A derived point shall be called a singular point or an apex of $f = 0$ according as it is or is not on $f = 0$. Apices do not arise if n is odd, since the left member of Euler's relation

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf$$

is zero at a derived point and therefore also f is zero. But if n is even, a derived point may not be on $f = 0$ and thus be an apex.

For example, any non-degenerate conic modulo 2 can be transformed linearly into $x^2 + yz = 0$. Its single derived point (100) is an apex.

Quartic curves modulo 2 have the remarkable property of possessing at most seven bitangents (or an infinity in a special case), whereas an algebraic quartic curve possesses twenty-eight in general. For the special quartic β of § 4, any line through the apex (001) is a bitangent, just as any line through the apex of the conic $x^2 + yz = 0$ is a tangent.

The number of non-equivalent types of quartic curves containing 0, 7, 6 real points and having no real linear factor is 8, 1, 6, respectively. In each case, the types are completely distinguished by the number and reality of the singular points and apices. Except for two types, in which there are only two bitangents and only two derived points, the intersections of the bitangents coincide completely with the derived points. The problem is more complicated in the case of quartic curves with five real points, there being twenty-five types (§ 7). Quartics with 1, 2, 3, or 4 real points have not been treated since they would probably not present sufficient novelty to compensate for the increased length of the investigation.

2. **Bitangents.** The general quartic is

$$Q = ax^4 + by^4 + cz^4 + dx^3y + exy^3 + fx^3z + gxz^3 + hy^3z + iyz^3 + jx^2y^2 + kx^2z^2 + ly^2z^2 + mx^2yz + nxy^2z + pxyz^2.$$

* Presented to the Society (San Francisco Section), October 24, 1914.

Either (I) $d \equiv e \equiv f \equiv g \equiv h \equiv i \equiv 0 \pmod 2$ or, after permuting the variables, we may assume that $d \equiv 1 \pmod 2$. In the latter case we replace x by $x + ey$ and obtain a Q with $d = 1$, $e = 0$. Replacing y by $y + fz$, and x by $x + mz$, we have also $f = m = 0$. In this case (II), $d = 1$, $e = f = m = 0$, the line $x = ry + sz$ is a bitangent to Q if the quartic in y and z obtained by eliminating x is a perfect square, i. e., has no terms in $y^3 z$ and yz^3; the conditions are

$$r^2 s + nr + h = 0, \qquad s^3 + ps + gr + i = 0.$$

If $g \equiv 1 \pmod 2$, there are at most 7 sets of solutions r, s. Next, let $g \equiv 0 \pmod 2$. Any root $\neq 0$ of the cubic in s leads to at most two r's. For $i = 0$ the root $s = 0$ leads to an infinitude of r's if and only if $n = h = 0$. In this case (II), $y = \rho z$ is a bitangent if and only if $\rho = 0$, $g = 0$, while $z = 0$ is not a bitangent. *There are at most seven bitangents in case* (II), *unless* $g = i = h = n = 0$, *when there is an infinitude of bitangents, viz., all lines through* (001).

For case (I), $z = 0$ is a bitangent; also $y = \rho z$ if and only if $n\rho^2 + p\rho = 0$; while $x = ry + sz$ is a bitangent only when

$$mr^2 + nr = 0, \qquad ms^2 + ps = 0.$$

In case (I), *there is an infinity of bitangents if two of the coefficients* m, n, p *are zero, otherwise at most seven.*

3. **Real Points on the Quartic.** The seven real points (i. e., with integral coördinates) modulo 2 are

$$1 = (100), \quad 2 = (010), \quad 3 = (001), \quad 4 = (110), \quad 5 = (101),$$
$$6 = (011), \quad 7 = (111).$$

The values of Q at these points are respectively

$$a, b, c, \quad a + b + d + e + j, \quad a + c + f + g + k, \quad b + c + h + i + l,$$

and the sum of the fifteen coefficients.

4. **Quartic Curves Without Real Points.** The seven values in § 3 are now unity, so that

$$a = b = c = 1, \qquad j = d + e + 1, \qquad k = f + g + 1, \qquad l = h + i + 1,$$
$$p = m + n + 1.$$

(I) Consider case (I) of § 2. If m and n are not both zero, we may set $m = 1$, after interchanging x and y if necessary. If $m = n = 1$, we get

$$\alpha = x^4 + y^4 + z^4 + x^2 y^2 + x^2 z^2 + y^2 z^2 + x^2 yz + xy^2 z + xyz^2,$$

which is invariant under all real linear transformations. Its bitangents are the seven real lines. Their intersections, the seven real points, are apices and give all the apices. There is no singular point.

If $m = 1, n = 0$, we interchange x and z and have the case $m = n = 0$, viz.,

$$\beta = x^4 + y^4 + z^4 + x^2y^2 + x^2z^2 + y^2z^2 + xyz^2.$$

The only bitangents are $y = 0$, $z = 0$ and $x = ry$, where r is arbitrary. Their intersections are 1, 3, and $(r\,1\,0)$, all being apices except the last for $r^2 + r + 1 = 0$, when $(r\,1\,0)$ is a singular point. There are no further derived points. Any line through apex 3 is a bitangent.

(II) Consider case (II) of § 2, viz., $d = 1, e = f = m = 0$.

(1) First, let $h = 1$. If $n = 1$, we replace y by $y + x$ and then x by $x + z$ and obtain a quartic with $h = 1, n = 0$, viz.,

$$\gamma_{g,i} = x^4 + y^4 + z^4 + x^3y + gxz^3 + (g+1)x^2z^2 + y^3z + iyz^3 + iy^2z^2 + xyz^2.$$

By § 2, the bitangents are $y = 0$ (if $g = 0$) and

$$x = ry + sz, \qquad r^2s = 1, \qquad s^3 + s + gr + i = 0.$$

If $g = i = 0$, the bitangents are $y = 0$ and $x = y + z$, crossing at the apex 5; the only further derived point is the apex 3.

If $g = 0$, $i = 1$, the bitangents are $y = 0$, $x = (s+1)y + sy$, where $s^3 + s + 1 = 0$; they intersect at the apices 7 and $(s01)$. There is no further derived point.

If $g = 1, i = 0$, the seven bitangents are

$$x = ry + r^{-2}z, \qquad r^7 + r^4 + 1 \equiv 0 \pmod 2.$$

Replacing r by another root r^2 of this irreducible congruence, we obtain a bitangent meeting the former at $P = (\eta^3\,\eta 1)$, where $\eta = r^6 + r^2 + 1$ is a root of $\eta^7 + \eta + 1 = 0$. The seven roots of the latter are obtained by replacing r by r^2 in succession in η, the resulting function, etc. These seven apices P are the only derived points.

If $g = i = 1$, the bitangents are

$$x = ry + r^{-2}z, \qquad r = 1, \qquad r^2 + r + 1 = 0 \quad \text{or} \quad r^4 + r^3 + 1 = 0.$$

Their intersections are seen to be the apex 6 and the singular points

$$(y^3 + y, y, 1), \qquad y^2 + y + 1 = 0, \quad \text{or} \quad y^4 + y + 1 = 0.$$

There are no further derived points.

(2) Second, let $h = g = 0$. If $n = 1$, $i = 0$, we replace x, y, z by y, $x + z$, x and then x by $x + y$ and get γ_{00}. If $n = i = 0$, we replace z by $z + x$ and have case (I). If $n = 0$, $i = 1$, we replace x by $x + y$, z by $z + y$, and get γ_{01}. If $n = i = 1$, we have

$$\delta = x^4 + y^4 + z^4 + x^3 y + x^2 z^2 + yz^3 + xy^2 z.$$

By § 2, its bitangents are $y = 0$, $x = z$, $x = sz$, $x = y + z$, $x = s^{-1} y + sz$, where $s^2 + s + 1 = 0$. Their intersections are the apices 2, 5, 7 and the singular points $(s01)$, $(s1s^2)$. There are no further derived points.

(3) Finally, let $h = 0$, $g = 1$. If $i = 0$, we apply (xyz) and, in case $n = 0$, also $x' = x + z$, and obtain case (1). Hence let $i = 1$. Then, if $n = 0$, we replace y by $y + x$ and have case (2). But if $n = 1$, we have

$$\epsilon = x^4 + y^4 + z^4 + x^3 y + xz^3 + yz^3 + xy^2 z.$$

By § 2, its bitangents are $x = z$, $x = sz$, $x = \sigma^{-1} y + \sigma z$, where

$$s^2 + s + 1 = 0, \qquad \sigma^4 + \sigma + 1 = 0.$$

Their intersections are the apex 2 and the singular points $(1\tau 1)$ and $(\tau\sigma^3 1)$ where $\tau = \sigma + \sigma^2$ is a root of $\tau^2 + \tau + 1 = 0$. There are no further derived points.

The eight classes of quartic curves modulo 2 without real points are distinguished by the number and reality of their apices and singular points. These derived points coincide with the intersections of the bitangents, except for γ_{00} which has only two bitangents and two apices.

5. **Quartic Curves Containing all Seven Real Points.** The conditions are

$$a = b = c = 0, \qquad j = d + e, \qquad k = f + g, \qquad l = h + i, \qquad p = m + n.$$

For case (I) of § 2, Q is the product of four linear functions. For case (II),

$$Q = x^3 y + x^2 y^2 + gxz^3 + gx^2 z^2 + hy^3 z + iyz^3 + (h + i) y^2 z^2 + nxy^2 z + nxyz^2.$$

It has the factor x if $h = i = 0$, factor y if $g = 0$, factor $x + y$ if $i = 1$, $h = n$, factor $x + z$ if $h = 0, n + i = 1$, factor $x + y + z$ if $n = i, h + i = 1$. We exclude these cases. Then, if $n = 0$, we have $h = i = 1$ and get

$$A = x^3 y + x^2 y^2 + xz^3 + x^2 z^2 + y^3 z + yz^3.$$

But, if $n = 1$, we have $h = 1$, $i = 0$; the resulting Q is derived from A by replacing x by $y + z$, y by x, z by y. The bitangents to A are

$$x = (s^3 + 1) y + sz, \qquad s^7 + s + 1 = 0.$$

Their intersections are the apices $(1z^3 z)$, $z^7 + z^3 + 1 = 0$, which give all the derived points. *The single type of irreducible quartic curve with seven real points has seven imaginary bitangents whose seven intersections are apices and give all the derived points.*

6. **Quartic Curves with Six Real Points.** After applying a real transformation, we may assume that the curve contains the real points other than 1. Then

$a=1, \quad b=c=0, \quad d+e+j=f+g+k=m+n+p=1, \quad l=h+i.$

(1) First, let d and f be not both zero. As in § 2, we may set $d=1$, $e=f=m=0$. Then Q has the factor x if $h=i=0$, factor $x+y$ if $h=n$, $g=i$, factor $x+z$ if $h=0$, $n=i$, factor $x+y+z$ if $h=n+1$, $g=i+n$. We exclude these cases. Then if $h=0$, we have $i=n=g=0$, and get

$$B=x^4+x^3y+x^2z^2+yz^3+y^2z^2+xyz^2.$$

Its four bitangents are $y=0$ and $x=sz$, where $s^3+s+1=0$. Their intersections are the singular point 2 and the apices $(s01)$. There are no further derived points.

Next, if $h=n=1$, we have $i=g+1$. According as $g=0$ or $g=1$, we get

$$C=x^4+x^3y+x^2z^2+y^3z+yz^3+xy^2z,$$

$$D=x^4+x^3y+xz^3+y^3z+y^2z^2+xy^2z.$$

The seven bitangents to C are $y=0$, $x=ry+z$, where $r^2+r+1=0$, and

$$x=\rho y+(\rho+\rho^3)z, \qquad \rho^4+\rho^3+1=0.$$

Their intersections are the singular point 5 and the apices $(r01)$, and $(x1x^2)$ where $x^4+x+1=0$. There are no further derived points.

The seven bitangents to D are $x=s^3y+sz$, $s^7+s^3+1=0$. This meets the one with s replaced by the root s^2 in $P=(x1z)$, where $x=s^5+s^4$, $z=s^4+s^3+s^2$, so that $z=x^3$, $x^7+x+1=0$. All seven roots of the latter are obtained by replacing s by s^2 in succession in x, the resulting function, etc. These seven apices P give all the derived points of D.

Finally, if $h=1$, $n=0$, we have $i=g+1$. Then if $g=0$ the function is derived from B by the transformation $x'=x+y$, $z'=z+y$. If $g=1$, it is derived from D by $z'=z+y$.

(2) Second, let $d=f=0$, while e and g are not both zero. Interchanging y and z, if necessary, we may set $e=1$. We make $h=n=0$ by use of $x'=x+z$ and $y'=y+z$. We may set $i=1$, $g=m$, since otherwise Q has the factor x, $x+y$, or $x+y+z$. According as $m=0$ or $m=1$, we have

$$E=x^4+xy^3+x^2z^2+yz^3+y^2z^2+xyz^2,$$

$$F=x^4+xy^3+xz^3+yz^3+y^2z^2+x^2yz.$$

The only bitangents to E are $y=0$ and $y=z$. They intersect at the apex 1. The only further derived point is the singular point 5.

The seven bitangents to F are $y=\alpha z$, where $\alpha^3=1$, and

$$x=ry+r^2z, \qquad r^4+r+1=0.$$